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Table of Contents: Volume 9 Number 9, 16 May, 2014

ARTICLES

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Full Length Research Paper

Properties of Bertrand curves in dual space

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Starting from ideas and results given by Özkaldi, İlarslan and Yaylı in (2009), in this paper we investigate Bertrand curves in three dimensional dual space D^3. We obtain the necessary characterizations of these curves in dual space D^3. As a result, we find that the distance between two Bertrand curves and the dual angle between their tangent vectors are constant. Also, well known characteristic property of Bertrand curve in Euclid space E^3 which is the linear relation between its curvature and torsion is satisfied in dual space as $\hat{\lambda}.\hat{\kappa}(s)+\hat{\mu}.\hat{\tau}(s)=1$. We show that involute curves, which are the curves whose tangent vectors are perpendicular, of a curve constitute Bertrand pair curves.

Key words: Bertrand curves, involute-evolute curves, dual space.

2000 Mathematics Subject Classification: 53A04 , 53A25 , 53A40.

INTRODUCTİON

In the study of differential geometry, the characterizations of the curves and the corresponding relations between the curves are significant problems. It is well known that many important results in the theory of the curves in E^3 were given by G. Monge and then G. Darboux detected the idea of moving frame. After this, Frenet defined moving frame and special equations which are used in mechanics, kinematics and physics.

A set of orthogonal unit vectors can be built, if a curve is differentiable in an open interval, at each point. These unit vectors are called Frenet frame. The Frenet vectors along the curve, define curvature and torsion of the curve. The frame vectors, curvature and torsion of a curve constitute Frenet apparatus of the curve.

It is certainly well known that a curve can be explained by its curvature and torsion except as to its position in space. The curvature (κ) and torsion (τ) of a regular curve help us to specify the shape and size of the curve. Such as If $\kappa=\tau=0$, then the curve is a geodesic. If $\kappa\neq 0$ (constant) and $\tau=0$, then the curve is a circle with radius $\frac{1}{\kappa}$. If $\kappa\neq 0$ (constant) and $\tau\neq 0$ (constant), then the curve is a helix (Guggenheimer, 1963; Struik, 1988).

Bertrand curves can be given as another example of that relation. Bertrand curves are discovered in 1850, by J. Bertrand who is known for his applications of differential equations to physics, especially thermodynamics. A Bertrand curve in E^3 is a curve such that its principal normal vectors are the principal normal vectors of another curve. It is proved in most studies on the subject that the characteristic property of a Bertrand

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curve is the existence of a linear relation between its curvature and torsion as:

$$\lambda\kappa + \mu\tau = 1$$

with constants $\lambda,\ \mu$ where $\lambda \neq 0$ (Kühnel, 2006).

Dual numbers were defined by Clifford (1849, 1879). After him E. Study used dual numbers and dual vectors to clarify a mapping from dual unit sphere to three dimensional Euclidean space E^3. This mapping is called Study mapping. Study mapping corresponds the dual points of a dual unit sphere to the oriented lines in E^3. So the set of oriented lines in Euclidean space E^3 is one to one correspondence with the points of dual space in $\mathbb{D}^3$.

In this paper, we study Bertrand curves in dual space $\mathbb{D}^3$.

PRELİMİNARİES

We now recall some basic notions about dual space and apparatus of curves.

The set $\mathbb{D}$ is called the dual number system and the elements of this set are in type of $\widehat{a} = a + \varepsilon a^*$. Here a and a^* are real numbers and $\varepsilon^2 = 0$ which is called a dual unit. The elements of the set $\mathbb{D}$ are called dual numbers. The set $\mathbb{D}$ is given by

$$\mathbb{D} = \{\widehat{a} = a + \varepsilon a^* \mid a, a^* \epsilon\ \mathbb{R}\}.$$

For the dual number $\widehat{a} = a + \varepsilon a^*$, $a \in \mathbb{R}$ is called the real part of $\widehat{a}$ and $a^* \in \mathbb{R}$ is called the dual part of $\widehat{a}$. Two inner operations and equality on $\mathbb{D}$ are defined for $\widehat{a} = a + \varepsilon a^*$ and $\widehat{b} = b + \varepsilon b^*$ as ;

1) $+ : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$

$$\widehat{a} + \widehat{b} = (a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*)$$

is called the addition in $\mathbb{D}$.

2) $\cdot : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$

$$\widehat{a} \cdot \widehat{b} = (a + \varepsilon a^*).(b + \varepsilon b^*) = a.b + \varepsilon(ab^* + ba^*)$$

is called the multiplication in $\mathbb{D}$.

3) $\widehat{a} = \widehat{b}$ if and only if $a = b$ and $a^* = b^*$. (Köse et al., 1988; Veldkamp, 1976)

Also, the set $\mathbb{D} = \{\widehat{a} = a + \varepsilon a^* \mid a, a^* \epsilon\ \mathbb{R}\}$ forms a commutative ring with the following operations

$$i)\ (a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*)$$

$$ii)\ (a + \varepsilon a^*).(b + \varepsilon b^*) = a.b + \varepsilon(ab^* + ba^*).$$

The division of two dual numbers $\widehat{a} = a + \varepsilon a^*$ and $\widehat{b} = b + \varepsilon b^*$ provided $b \neq 0$ can be defined as

$$\frac{\widehat{a}}{\widehat{b}} = \frac{a + \varepsilon a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon \frac{a^* b - a b^*}{b^2}.$$

The set

$$\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \left\{ \begin{array}{c} \vec{\widehat{a}} \mid \vec{\widehat{a}} = (a_1 + \varepsilon a_1^*, a_2 + \varepsilon a_2^*, a_3 + \varepsilon a_3^*) \\ = (a_1, a_2, a_3) + \varepsilon (a_1^*, a_2^*, a_3^*) \\ = \vec{a} + \varepsilon \vec{a^*}, \quad \vec{a} \in \mathbb{R}^3,\ \vec{a^*} \in \mathbb{R}^3 \end{array} \right\}$$

is a module on the ring $\mathbb{D}$ which is called $\mathbb{D}$. Module and the elements are dual vectors consisting of two real vectors (Çöken and Görgülü, 2002; Güven et al., 2011).

The inner product and vector product of $\vec{\widehat{a}} = \vec{a} + \varepsilon \vec{a^*} \in \mathbb{D}^3$ and $\vec{\widehat{b}} = \vec{b} + \varepsilon \vec{b^*} \in \mathbb{D}^3$ are given by

$$\begin{aligned} \left\langle \vec{\widehat{a}}, \vec{\widehat{b}} \right\rangle &= \left\langle \vec{a}, \vec{b} \right\rangle + \varepsilon \left(\left\langle \vec{a}, \vec{b^*} \right\rangle + \left\langle \vec{a^*}, \vec{b} \right\rangle \right) \\ \vec{\widehat{a}} \times \vec{\widehat{b}} &= \left(\widehat{a}_2 \widehat{b}_3 - \widehat{a}_3 \widehat{b}_2\ , \widehat{a}_3 \widehat{b}_1 - \widehat{a}_1 \widehat{b}_3\ , \widehat{a}_1 \widehat{b}_2 - \widehat{a}_2 \widehat{b}_1 \right) \end{aligned}$$

where $\widehat{a}_i = a_i + \varepsilon a_i^*\ , \widehat{b}_i = b_i + \varepsilon b_i^* \in \mathbb{D}\ , 1 \leq i \leq 3.$

The norm $\left\| \vec{\widehat{a}} \right\|$ of $\vec{\widehat{a}}$ is defined by

$$\left\| \vec{\widehat{a}} \right\| = \sqrt{\left\langle \vec{\widehat{a}}, \vec{\widehat{a}} \right\rangle} = \|\vec{a}\| + \varepsilon \frac{\left\langle \vec{a}, \vec{a^*} \right\rangle}{\|\vec{a}\|}$$

where $a \neq 0$.

If the norm of $\vec{\widehat{a}}$ is 1, then $\vec{\widehat{a}}$ is called a dual unit vector.

Let

$$\begin{aligned} \widehat{\alpha} : I \subset \mathbb{D} &\longrightarrow \mathbb{D}^3 \\ \lambda &\longrightarrow \vec{\widehat{\alpha}}(\lambda) = \vec{\alpha}(\lambda) + \varepsilon \vec{\alpha^*}(\lambda) \end{aligned}$$

be a dual space curve with differentiable vectors $\vec{\alpha}(\lambda)$ and $\vec{\alpha^*}(\lambda)$. The dual arc-length parameter of $\vec{\widehat{\alpha}}(\lambda)$ is defined as

$$s = \int_{t_1}^{t} \left\| \overrightarrow{\widehat{\alpha}}(\lambda)' \right\| d\lambda.$$

Now we will give dual Frenet vectors of the dual curve

$$\begin{array}{cccc} \widehat{\alpha} : I \subset \mathbb{D} & \longrightarrow & \mathbb{D}^3 \\ s & \longrightarrow & \overrightarrow{\widehat{\alpha}}(s) = \overrightarrow{\alpha}(s) + \varepsilon \overrightarrow{\alpha^*}(s) \end{array}$$

with the dual arc-length parameter $s.$ Then

$$\frac{d\overrightarrow{\widehat{\alpha}}}{d\widehat{s}} = \frac{d\overrightarrow{\widehat{\alpha}}}{ds} \cdot \frac{ds}{d\widehat{s}} = \overrightarrow{\widehat{T}}$$

is called the unit tangent vector of $\overrightarrow{\widehat{\alpha}}(s)$. The norm of the vector $\frac{d\overrightarrow{\widehat{T}}}{ds}$ which is given by

$$\frac{d\overrightarrow{\widehat{T}}}{d\widehat{s}} = \frac{d\overrightarrow{\widehat{T}}}{ds} \cdot \frac{ds}{d\widehat{s}} = \frac{d^2\overrightarrow{\widehat{\alpha}}}{d\widehat{s}^2} = \widehat{\kappa}\overrightarrow{\widehat{N}}$$

is called curvature function of $\overrightarrow{\widehat{\alpha}}(s)$. Here $\widehat{\kappa} : I \longrightarrow \mathbb{D}$ is nowhere pure-dual. Then the unit principal normal vector of $\overrightarrow{\widehat{\alpha}}(s)$ is defined as

$$\overrightarrow{\widehat{N}} = \frac{1}{\widehat{\kappa}} \cdot \frac{d\overrightarrow{\widehat{T}}}{d\widehat{s}}$$

The vector $\overrightarrow{\widehat{B}} = \overrightarrow{\widehat{T}} \times \overrightarrow{\widehat{N}}$ is called the binormal vector of $\overrightarrow{\widehat{\alpha}}(s)$. Also, we call the vectors $\overrightarrow{\widehat{T}}, \overrightarrow{\widehat{N}}, \overrightarrow{\widehat{B}}$ Frenet trihedron of $\overrightarrow{\widehat{\alpha}}(s)$ at the point $\widehat{\alpha}(s)$. The derivatives of dual Frenet vectors $\overrightarrow{\widehat{T}}, \overrightarrow{\widehat{N}}, \overrightarrow{\widehat{B}}$ can be written in matrix form as:

$$\begin{bmatrix} \overrightarrow{\widehat{T}}' \\ \overrightarrow{\widehat{N}}' \\ \overrightarrow{\widehat{B}}' \end{bmatrix} = \begin{bmatrix} 0 & \widehat{\kappa} & 0 \\ -\widehat{\kappa} & 0 & \widehat{\tau} \\ 0 & -\widehat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{\widehat{T}} \\ \overrightarrow{\widehat{N}} \\ \overrightarrow{\widehat{B}} \end{bmatrix}$$

which are called Frenet formulas (Köse et al. (1988). The function $\widehat{\tau} : I \longrightarrow \mathbb{D}$ such that $\frac{d\overrightarrow{\widehat{B}}}{d\widehat{s}} = -\widehat{\tau}\overrightarrow{\widehat{N}}$ is called the torsion of $\overrightarrow{\widehat{\alpha}}(s)$ which is nowhere pure-dual.

For a general parameter t of a dual space curve $\overrightarrow{\widehat{\alpha}}$, the curvature and torsion of $\overrightarrow{\widehat{\alpha}}$ can be calculated as:

$$\widehat{\kappa} = \frac{\left\| \widehat{\alpha}' \times \widehat{\alpha}'' \right\|}{\left\| \widehat{\alpha}' \right\|^3} \quad , \quad \widehat{\tau} = \frac{\det\left(\widehat{\alpha}', \widehat{\alpha}'', \widehat{\alpha}'''\right)}{\left\| \widehat{\alpha}' \times \widehat{\alpha}'' \right\|}.$$

BERTRAND CURVES IN D^3

Here, we define Bertrand curves in dual space D^3 and give characterizations and theorems for these curves.

Definition 1

Let D^3 be the dual space with standard inner product $\langle , \rangle$ and $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ be the dual space curves. If there exists a corresponding relationship between the dual space curves $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ so that the principal normal vectors of $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ are linear dependent to each other at the corresponding points of the dual curves, then $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ are called Bertrand curves in D^3.

Let the curves $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ be Bertrand curves in D^3, parameterized by their arc-length s and $\widehat{s},$ respectively. Let $\left\{\overrightarrow{\widehat{T}}(s), \overrightarrow{\widehat{N}}(s), \overrightarrow{\widehat{B}}(s)\right\}$ indicate the unit Frenet frame along $\overrightarrow{\widehat{\alpha}}$ and $\left\{\overrightarrow{\widehat{T}^{\circ}}(s), \overrightarrow{\widehat{N}^{\circ}}(s), \overrightarrow{\widehat{B}^{\circ}}(s)\right\}$ the unit Frenet frame along $\overrightarrow{\widehat{\beta}}$. Also $\widehat{\kappa}(s) = \kappa(s) + \varepsilon\kappa^*(s)$ and $\widehat{\tau}(s) = \tau(s) + \varepsilon\tau^*(s)$ are the curvature and torsion of $\overrightarrow{\widehat{\alpha}}$, respectively. Similarly, $\widehat{\kappa^{\circ}}(s) = \kappa^{\circ}(s) + \varepsilon\kappa^{\circ *}(s)$ and $\widehat{\tau^{\circ}}(s) = \tau^{\circ}(s) + \varepsilon\tau^{\circ *}(s)$ are the curvature and torsion of $\overrightarrow{\widehat{\beta}}$, respectively.

In the following theorems, we obtain the characterizations of a dual Bertrand curve.

Theorem 1

Let $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ be two curves in D^3. If $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ are Bertrand curves, then

$$d\left(\overrightarrow{\widehat{\alpha}}(s), \overrightarrow{\widehat{\beta}}(s)\right) = \widehat{c}$$

where $s \in I \subset \mathrm{D}$ and $\widehat{c} \in \mathbb{D}$ (constant).

Proof: Let $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ be Bertrand curves.

If $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ are Bertrand curves , then it can be written from Figure 1;

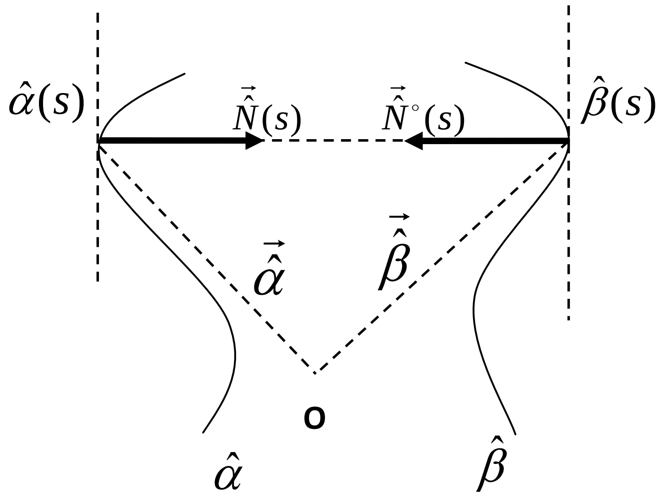

Figure 1. Bertrand pair curves.

$$\vec{\hat{\beta}}(s) = \vec{\hat{\alpha}}(s) + \hat{\lambda}(s)\vec{\hat{N}}(s) \tag{1}$$

for the vectors of $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$

By differentiating the Equation 1 with respect to s and applying the Frenet formulas,

$$\frac{d\hat{s}}{ds}.\vec{\hat{T}^{\circ}}(s) = \left(1 - \hat{\lambda}(s)\hat{\kappa}(s)\right).\vec{\hat{T}}(s) + \hat{\lambda}'(s).\vec{\hat{N}}(s) + \hat{\lambda}(s).\hat{\tau}(s).\vec{\hat{B}}(s). \tag{2}$$

is obtained.

If we take the inner product of the Equation 2 with $\vec{\hat{N}}(s)$ both sides,

$$0 = \hat{\lambda}'(s)$$

is found.Thus, we notice that

$$\hat{\lambda}(s) = \hat{c}$$

where $\hat{c} \in \mathbb{D}$ (constant). If we use

$$d\left(\vec{\hat{\alpha}}(s), \vec{\hat{\beta}}(s)\right) = \left\|\vec{\hat{\beta}}(s) - \vec{\hat{\alpha}}(s)\right\|$$

and the Equation 1 we get

$$d\left(\vec{\hat{\alpha}}(s), \vec{\hat{\beta}}(s)\right) = \hat{c}$$

where $s \in I \subset \mathrm{D}$ and $\hat{c} \in \mathbb{D}$ (constant).

Theorem 2

Let $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ be two curves in D^3. If $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ are Bertrand curves, then the dual angle between the tangent vectors at the corresponding points of the dual Bertrand curves is constant.

Proof: Let $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ be two Bertrand curves in D^3.

If the dual angle between the tangent vectors $\vec{\hat{T}}(s)$ and $\vec{\hat{T}^{\circ}}(s)$ at the corresponding points of $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ is

$$\phi = \varphi + \varepsilon\varphi^* \in \mathrm{D},$$

then we write

$$\vec{\hat{T}^{\circ}}(s) = \cos\phi\,\vec{\hat{T}}(s) + \sin\phi\,\vec{\hat{B}}(s). \tag{3}$$

If we differentiate the last equation of the above, we obtain

$$\hat{\kappa}^{\circ}(s)\vec{\hat{N}^{\circ}}(s)\frac{d\hat{s}}{ds} = \frac{d\cos\phi}{ds}\vec{\hat{T}}(s) + \left(\hat{\kappa}(s)\cos\phi - \hat{\tau}(s)\sin\phi\right)\vec{\hat{N}}(s) + \frac{d\sin\phi}{ds}\vec{\hat{B}}(s). \tag{4}$$

If we take the inner product of the last equation of the above with $\vec{\hat{T}}(s)$ both sides and we use the Frenet formulas, we have

$$\frac{d\cos\phi}{ds} = 0.$$

So

$$\cos\phi = \text{constant}$$

is found where $\phi = \varphi + \varepsilon\varphi^* \in \mathrm{D}$.

This completes the proof.

Conclusion 1

If $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ be dual Bertrand curves, then the distance between the corresponding points of them and the dual angle between the tangent vectors are constant.

Theorem 3

Let $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ be two curves in D^3. If $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ are Bertrand curves, $\hat{\kappa}(s)$ and $\hat{\tau}(s)$ are the curvature and

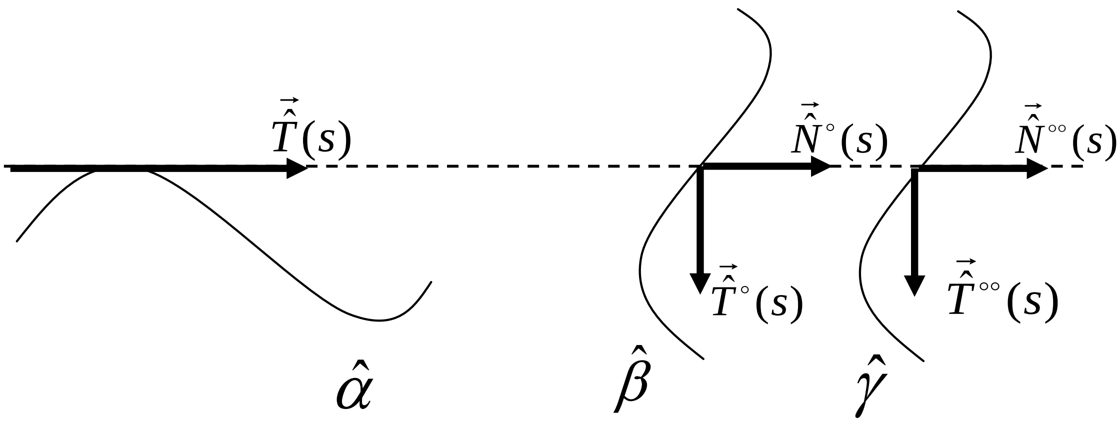

Figure 2. Involute curves of $\widehat{\alpha}$.

torsion of $\overrightarrow{\widehat{\alpha}}$, $\widehat{\kappa^{\circ}}(s)$ and $\widehat{\tau^{\circ}}(s)$ are the curvature and torsion of $\overrightarrow{\widehat{\beta}}$, respectively, then

$$\widehat{\lambda}.\widehat{\kappa}(s)+\widehat{\mu}.\widehat{\tau}(s)=1 \qquad (5)$$

where $\widehat{\lambda},\widehat{\mu}\in\mathbb{D}$ are constant.

Proof: Let $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ are Bertrand curves.

If the dual angle between the tangent vectors $\overrightarrow{\widehat{T}}(s)$ and $\overrightarrow{\widehat{T^{\circ}}}(s)$ at the corresponding points of $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ is

$\phi=\varphi+\varepsilon\varphi^{*}\in$ D,

then from the previous proof we have

$$\overrightarrow{\widehat{T^{\circ}}}(s)=\cos\phi\,\overrightarrow{\widehat{T}}(s)+\sin\phi\,\overrightarrow{\widehat{B}}(s) \qquad (6)$$

From the Equation (2) we write

$$\frac{d\widehat{s}}{ds}.\overrightarrow{\widehat{T^{\circ}}}(s)=\left(1-\widehat{\lambda}.\widehat{\kappa}(s)\right).\overrightarrow{\widehat{T}}(s)+\widehat{\lambda}.\widehat{\tau}(s).\overrightarrow{\widehat{B}}(s). \qquad (7)$$

In above equations, if we take into account $\frac{d\widehat{s}}{ds}=\widehat{a}$ (constant)

then we get

$$1-\widehat{\lambda}.\widehat{\kappa}(s)=\cot\phi.\widehat{\lambda}.\widehat{\tau}(s). \qquad (8)$$

We can write

$$\widehat{\mu}=\widehat{\lambda}.\cot\phi \text{ (constant).} \qquad (9)$$

Finally, from the Equations 8 and 9, we find

$$\widehat{\lambda}.\widehat{\kappa}(s)+\widehat{\mu}.\widehat{\tau}(s)=1.$$

Conclusion 2

If $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ be dual Bertrand curves, then the characterization of being Bertrand curve in dual space is same with Euclid space.

Theorem 4

Let $\overrightarrow{\widehat{\alpha}}$ be a plane curve in D^3. If $\overrightarrow{\widehat{\beta}}$ and $\overrightarrow{\widehat{\gamma}}$ are the involutes of $\overrightarrow{\widehat{\alpha}}$, then $\overrightarrow{\widehat{\beta}}$ and $\overrightarrow{\widehat{\gamma}}$ are Bertrand curves in D^3.

Proof: Let $\overrightarrow{\widehat{\beta}}$ and $\overrightarrow{\widehat{\gamma}}$ be the involutes of $\overrightarrow{\widehat{\alpha}}$ as in Figure 2.

It can be written from Caliskan et al. (in press)

$$\widehat{\tau^{\circ}}(s)=\frac{\widehat{\kappa}(s).\widehat{\tau}'(s)-\widehat{\kappa}'(s).\widehat{\tau}(s)}{\widehat{\kappa}(s).(\widehat{c}-s).\left(\widehat{\kappa}^{2}(s)+\widehat{\tau}^{2}(s)\right)} \qquad (10)$$

where $\widehat{\kappa}(s)$, $\widehat{\tau}(s)$ and $\widehat{\kappa^{\circ}}(s)$, $\widehat{\tau^{\circ}}(s)$ are the curvature and torsion of $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$, respectively, and $\widehat{c}\in\mathbb{D}$ (constant).

If $\overrightarrow{\widehat{\alpha}}$ is a plane curve, then

$$\widehat{\tau}(s)=0. \qquad (11)$$

From the last two equation given above, we have

$$\widehat{\tau^{\circ}}(s) = 0.$$

So $\vec{\hat{\beta}}$ is a plane curve. Similarly, $\vec{\hat{\gamma}}$ is a plane curve. Consequently, the principal normal vectors of $\vec{\hat{\beta}}$ and $\vec{\hat{\gamma}}$ are linearly dependent to each other at the corresponding points of the dual curves. In that case $\vec{\hat{\beta}}$ and $\vec{\hat{\gamma}}$ curves are Bertrand curves in D^3.

Conflict of Interest

The author(s) have not declared any conflict of interests.

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Angular symmetry of space-time and the spinor representation of Poincaré group

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In the paper, a relativistic theory is suggested; we add three independent angles to the four coordinates of the Minkowski space that define the (x, t) - position of a moving local observer. These angular coordinates define the orientation of an observer under free rotations and they allow us to introduce the generators of the Poincaré group in the angular representation. Instead of a multi-component wave function of any spin-component wave function, one component wave function Ψ (and its Lorentz transform) is introduced, depending on the four coordinates of Minkowski space and three angular coordinates. Poincaré invariant first-order linear differential equations are derived. The matrix representation of the above operator equations based on appropriate angular is equivalent to Dirac and Maxwell equations. It is predicted and proven that the number of generations of leptons is three.

Key words: Spinor representation, generators of the Poincaré group, dimensions of Minkowski space.

INTRODUCTION

It is assumed that states with spin j and the corresponding equations describing these states possess the angular symmetry of fields and particles. Diverse formulations are used for spin and Dirac and Maxwell equations (Fushchich and Nikitin, 1994; Varadarajan, 1989); however, the dependence of angular was represented implicitly in previous formulations. Beginning with Kaluza-Klein, numerous compactified and unobservable dimensions were introduced to explain the nature of the four types of fundamental forces (electromagnetic, gravitational, strong, and weak forces). For instance, Rumer and Fet (1977) introduced frames of reference of free rotation at any point of space-time with variable metric. The position of these frames of reference is defined by angles. The equivalence between the Schrödinger operator equation and Heisenberg's matrix mechanics was proved in 1927 for operators depending on the time-space coordinates (Teschl, 2009). A similar construction (substitution matrices by operators) dealing only with the operators acting on the angular variables (which are introduced to explain the nature of spinors) is considered in the present paper.

Complete knowledge of free particle states and their behaviour can be obtained once all the unitary irreducible representations of the Poincaré group are found (Ohnuki, 1988). The relationship between the Lorentz group and Poincaré group in the angular representation and the equations for relativistic particles is focused on in this article, as well as the obtaining of generalized Lorentz group and Poincaré group.

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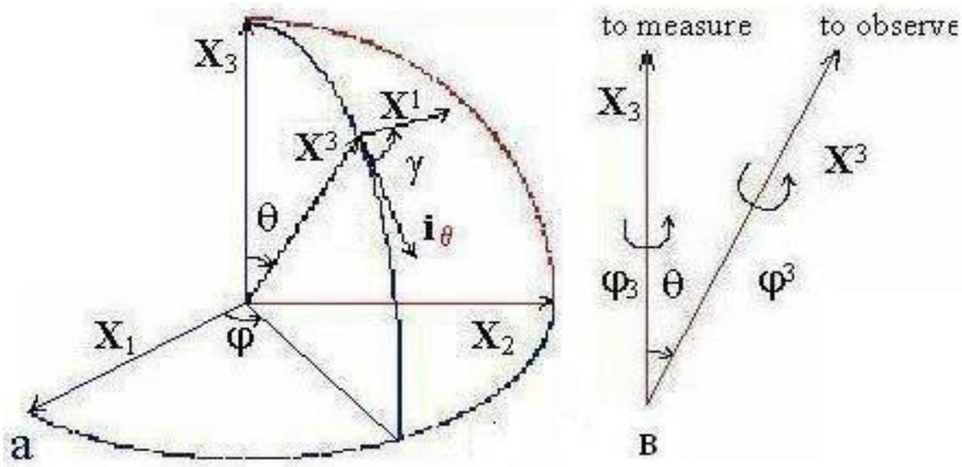

Figure 1. Orientation of the local observer in relation to the ordinary observer.

WAVE FUNCTION

The transformation properties of a multicomponent wave function that describes the transformation of fields and states with spin under rotations of ordinary observer or the original Cartesian of the coordinate system x_1, x_2, x_3 with unit vector $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are related to the existence of 'angular structure' for the spin states. At the center of the point x_1, x_2, x_3 of ordinary observer, we introduce a complementary local observer $\mathrm{X}^{(1)}, \mathrm{X}^{(2)}, \mathrm{X}^{(3)}$. Rotation of local observer does not change x_1, x_2, x_3, t. As a rule, a rotating local observer corresponds to a local (right) Cartesian system of coordinates $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}$ (the isovector space) centered at point x_1, x_2, x_3, with the orientation determined by the three independent Euler angles φ, θ, ν. The position and the orientation of the local observer taken with respect to ordinary observer are shown in Figure 1 and set as $\varphi_3 = \varphi, \varphi^{(3)} = \nu$.

Since the theory of relativity states it is equivalent to rotate the ordinary observer or the local observer (the state, object), then the transformation properties will be described from the point of view of the local observer.

In the Minkowski space, the states of spin particles or fields with spin can be described by using a one component wave function, depending on the position x_1, x_2, x_3, t and the orientation φ, θ, ν of a moving and free rotating local observer. All variables $x_1, x_2, x_3, t, \varphi, \theta, \nu$ are independent. φ, θ, ν are internal variables and fully describe the degree of freedom of the spin. Let $X^{(i)}{}_k = (\mathbf{X}^{(i)}\mathbf{X}_k)$ be the entries of the corresponding rotation matrix (or the projections of the corresponding unit vectors to one another) that has the following form (Biedenharn and Louck, 1984):

$$\mathrm{X}^{(3)}{}_3 = \cos(\theta), \mathrm{X}^{(1)}{}_1 = -\sin(\varphi)\sin(\nu) + \cos(\theta)\cos(\varphi)\cos(\nu),$$
$$\mathrm{X}^{(1)}{}_3 = -\sin(\theta)\cos(\nu), \mathrm{X}^{(1)}{}_2 = \cos(\varphi)\sin(\nu) + \cos(\theta)\sin(\varphi)\cos(\nu),$$
$$\mathrm{X}^{(2)}{}_3 = \sin(\theta)\sin(\nu), \mathrm{X}^{(2)}{}_1 = -\sin(\varphi)\cos(\nu) - \cos(\theta)\cos(\varphi)\sin(\nu),$$
$$\mathrm{X}^{(3)}{}_1 = \sin(\theta)\cos(\varphi), \mathrm{X}^{(2)}{}_2 = \cos(\varphi)\cos(\nu) - \cos(\theta)\sin(\varphi)\sin(\nu),$$
$$\mathrm{X}^{(3)}{}_2 = \sin(\theta)\sin(\varphi)$$

To the multicomponent wave function $C = (C_1, C_2, ..C_k)$ with the amplitudes $C_i(x_1, x_2, x_3, t)$, we assign the one-component wave function $\Psi = \psi C$. The set of the basis $\psi = (\psi_1, \psi_2, ..\psi_k)$ depends only on φ, θ, ν and $<\psi_n|\psi_i> = \delta_{ni}$. The expansion of Ψ with respect to the basis corresponds Ψ to the multicomponent wave function C, $C_i = <\Psi|\psi_i>$, and by the Sommerfeld condition, the wave function $\Psi = \psi C$ must be invariant under the Lorentz transformations. Obviously, the transformation properties of the basis Ψ and of the amplitudes C are dual. The basis of the states with the spin j, which consists of 2j+1 functions $\psi = (\psi_1, \psi_2, ..\psi_{2j+1})$ must have the following transformation properties (Biedenharn and Louck, 1984):

$$\psi' = \psi D^j(\alpha, \beta, c) \tag{1}$$

Where α, β, c stand for the angles defining the new orientation of the ordinary observer, the angles φ', θ', ν' define the new variables in the coordinate system of the ordinary observer, and the matrix of the Wigner functions (or the rotation matrix) D^j has the following properties (Biedenharn and Louck, 1984):

$$\overline{D}^j(\varphi', \theta', \nu') = D^j(\alpha, \beta, c)^T \overline{D}^j(\varphi, \theta, \nu) \tag{2}$$

Here, T stands for the transposition, the over line means complex conjugation, and S_k denotes the matrix operator of the spin angular momentum j in the z-representation. For example, the transformation (Equation 1) for j = 1/2 is of the form:

$$\begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = \cos(\omega/2)\sigma_0 - i\sin(\omega/2)\sigma_k \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{3}$$

Where ω stands for the rotation angle about the axis X_k and σ k are the Pauli matrices,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \sigma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

denoted by $\xi^{(k)}$, which is the column of the Wigner matrix $\overline{D}^{j}$ with an index $k = j, j-1,..,-j$. Let $\xi^{(k)}{}_{m} = \overline{D}^{j}{}_{m,k}$, the Wigner functions satisfy the normalization conditions

$$< \xi^{(m)}{}_{i} \left| \xi^{(k)}{}_{n} \right. >= \iiint \xi^{(m)}{}_{i} \overline{\xi}^{(k)}{}_{n} d\Omega d\nu = \delta_{mk}\delta_{in} / b^{2}$$

Where $d\Omega = \sin(\theta)d\theta\, d\varphi$ stands for the solid angle, δ_{ik} _is the Kronecker delta, $b = \sqrt{(2j+1)}/(4\pi)$ is the known normalizing coefficient, and the domains of the angles in the arguments of the Wigner D-function are equal to $0 \le \varphi \le 4\pi, 0 \le \theta \le \pi, 0 \le \nu \le 2\pi$, respectively. Moreover, $X^{(i)}{}_{k}$ stands for the basis with j = 1, and we also have $\mathbf{X}^{(\pm)} = \mathbf{X}^{(1)} \mp \mathbf{X}^{(2)}, < X^{(m)}{}_{i} \left| X^{(k)}{}_{n} \right. >= \delta_{mk}\delta_{in} / b^{2}$

Proposition 1

The basis of spinors with transformation properties of Equation 1 consists either of the columns $\xi^{(m)}$ or of their linear combinations.

Proof

Equation 2 for a column of the matrix D^{j} (Biedenharn and Louck, 1984) is equivalent to the relation $\xi^{(m)'} = D^{j}(\alpha,\beta,c)^{T}\xi^{(m)}$ or Equation 1.

Spherical harmonics $Y(\theta,\varphi)^{j}{}_{m} = \overline{D}^{j}(\varphi,\theta,\nu)_{m,0}\sqrt{(2j+1)}/\sqrt{(4\pi)}$, where j – integer is the eigen functions of the operators angular moment of L that depends only on two angles φ,θ. It is known that 2j+1 of spherical harmonics of $Y = (Y^{j}{}_{j}, Y^{j}{}_{j-1},..Y^{j}{}_{-j})$ will be transformed at rotation in space on a representation of a group of SO(3), coinciding with Equation 1, $Y' = YD^{j}(\alpha,\beta,c)$.

Spinor representation of a group of rotation is given by matrix $D^{(j)}$ dimension 2j + 1, the Lie algebra which is isomorphic to the Lie algebra of three-dimensional rotations SO(3), (Biedenharn and Louck, 1984). Quantities-(2j +1)-dimensional vector transformations according to the spinor representation of Equation 1 are called spinors. For matrices of even dimension, this corresponds to an irreducible representation unitary group SU(2). Irreducible unitary representations of group SO(3) always have odd dimension.

Theorem 1

Two spherical angles (φ stands for the azimuthal angle and θ for the polar angle) are insufficient to describe the transformation properties of state with spin by using the wave function $\Psi(\varphi,\theta)$. For non-integer j, there are no 2j + 1 functions $\psi = (\psi_{1}, \psi_{2},..\psi_{2j+1})$ of the variables φ,θ that are transformed according to some representation of the group SU(2) by (1).

Proof

Substituting $\xi^{(m)}(\varphi,\theta,\nu) = \exp(im\nu)\xi^{(m)}(\varphi,\theta,0)$ into Equation 1, we obtain

$$\xi^{(m)}(\varphi',\theta',0) = \exp(imf)D^{j}(\alpha,\beta,c)\xi^{(m)}(\varphi,\theta,0), f = \nu' - \nu.$$

This means that Relation 1 holds for $\xi^{(m)}(\varphi,\theta,0)$ up to the phase factor $\exp(imf)$. Further, let us define the dependence of the basis $\Psi_{\pm}$ of eigen functions S_{3} on φ,θ for j = 1/2. To this end, consider the rotation by the angle α around X_3. On the section of the sphere θ = const we have

$$\varphi' = \varphi - \alpha, \psi(\varphi - \alpha)_{\pm} = \exp(\mp\alpha/2)\psi(\varphi)_{\pm}.$$

Therefore, $\psi(\varphi)_{\pm} = \exp(\pm\varphi/2)\psi(0)_{\pm}$.

Let us now define the dependence of the basis $\eta_{\pm}$ of the eigen functions S_{2} for j = 1/2. The bases of the eigen functions S_{2} and S_{3} are connected in accordance with Equation 3, $\eta_{\pm} = \psi_{+} \mp \psi_{-}$. Making a rotation in the section of the sphere $\varphi = 0$ by the angle β around $\mathbf{X}_2$, we have

$$\theta' = \theta + \beta, \eta(\theta + \beta)_{\pm} = \exp(\mp\beta/2)\eta(\theta)_{\pm}.$$

Therefore, $\eta(\theta)_{\pm} = \exp(\mp i\theta/2)\eta(0)_{\pm}$. Moreover, we use the rotation around X_2 by the angle π, which is equivalent to the transformation $\widehat{G}: \varphi' = \pi - \varphi, \theta' = \pi - \theta$. Here, we have $\widehat{G}\psi_{\pm} = \pm\psi_{\mp}$. We finally obtain the form of a basis whose existence is assumed, namely, $(\psi_{+},\psi_{-})^{T} = C_{1}\xi^{(1/2)}(\varphi,\theta,0) + C_{2}\xi^{(-1/2)}(\varphi,\theta,0).$ This basis is transformed according to Equation 1 up to the phase factor exp(i f/2).

The generators of the rotation groups of the ordinary observer and local observer or of the vector and isovector rotations for the functions φ,θ,γ are the operators of angular momentum J_k, $J^{(n)}$ (Biedenharn and Louck, 1984).

$$J_{1} = i\cos(\varphi)ctg(\theta)\partial/\partial\varphi + i\sin(\varphi)\partial/\partial\theta - i\cos(\varphi)/\sin(\theta)\partial/\partial\nu,$$
$$J_{2} = i\sin(\varphi)ctg(\theta)\partial/\partial\varphi - i\cos(\varphi)\partial/\partial\theta - i\sin(\varphi)/\sin(\theta)\partial/\partial\nu,$$
$$J_{3} = -i\partial/\partial\varphi, J^{(3)} = -i\partial/\partial\nu,$$
$$J^{(1)} = -i\cos(\nu)ctg(\theta)\partial/\partial\nu - i\sin(\nu)\partial/\partial\theta + i\cos(\nu/\sin(\theta)\partial/\partial\varphi,$$
$$J^{(2)} = i\sin(\nu)ctg(\theta)\partial/\partial\nu - i\cos(\nu)\partial/\partial\theta - i\sin(\nu)/\sin(\theta)\partial/\partial\varphi.$$

The rotation groups of the ordinary observer and the local observer commute with each other and the spatial and isovector rotations are realized. To be more precise,

$$[J_{i}, J_{n}] = i\varepsilon_{ink}J_{k}, [J_{i}, J^{(n)}] = 0, [J^{(i)}, J^{(n)}] = -i\varepsilon_{ink}J^{(k)}, \tag{4}$$

Where ε_{ijk} stands for the antisymmetric tensor with $\varepsilon_{123} = 1$ and the eigen functions of the operators J^2, $J^{(3)}$ and J_3 are Wigner D-functions, $\mathbf{J}^{2} = \sum J_{k}J_{k}$ and

$$J_{3}\overline{D}^{j}{}_{m,m'} = m\overline{D}^{j}{}_{m,m'}, J^{(3)}\overline{D}^{j}{}_{m,m'} = m'\overline{D}^{j}{}_{m,m'}, J^{2}\overline{D}^{j}{}_{m,m'} = j(j+1)\overline{D}^{j}{}_{m,m'}.$$

The operators of raising and lowering indices m,m' act by the usual rules,

$$J_{+} = J_{1} + iJ_{2}, J^{(-)} = J^{(1)} + iJ^{(2)}, \overline{J}^{(-)} = -J^{(+)}, \overline{J}_{-} = -J_{+},$$
$$J_{+}\overline{D}^{j}{}_{m,m'} = \sqrt{(j-m)(j+m+1)}\overline{D}^{j}{}_{m+1,m'}, J^{(+)}\overline{D}^{j}{}_{m,m'} = \sqrt{(j-m')(j+m'+1)}\overline{D}^{j}{}_{m,m'+1}$$

and the columns $\overline{\mathrm{D}}^{j}$, j = 1/2 are the spinors $\xi^{(1/2)}=(\psi_1,\psi_2),\xi^{(-1/2)}=(\psi_3,\psi_4)$.

$$\psi_1=\cos(\theta/2)\exp(\mathrm{i}\nu/2+\mathrm{i}\varphi/2),\psi_3=-\sin(\theta/2)\exp(-\mathrm{i}\nu/2+\mathrm{i}\varphi/2),$$
$$\psi_2=\sin(\theta/2)\exp(\mathrm{i}\nu/2-\mathrm{i}\varphi/2),\psi_4=\cos(\theta/2)\exp(-\mathrm{i}\nu/2-\mathrm{i}\varphi/2), \quad (5)$$

In z-representation, the spin operator S_p and its eigenvectors are identical to the matrix representation J_p on the basis of the spinor $\psi=\xi^{(m)}$ for any index $m=j,j-1,..,-j$.

We refer to the operator J_p as the spin operator in the operator representation $J_p\psi=\psi S_p$. In terms of J_p, the transformation of Equation 1 acquires the form $\xi^{(m)\prime}=\exp(-i\omega J_i)\xi^{(m)}$, because the action of the rotation operator $\exp(-i\omega S_i)$ is identical to the operator $\exp(-i\omega J_i)$.

Proposition 2

The rotation of a spin state (spinor) by the angle π around the axis x_2 corresponds to the transformation $\widehat{G}_2=\exp(\text{-i}\,\pi J_2)$, which is equivalent to the angles $\widehat{\mathrm{G}}_2:\varphi'=\pi-\varphi,\theta'=\pi-\theta,\nu'=\nu-\pi$.

For $j=1/2,\widehat{G}_2\,\psi_1=\psi_2,\widehat{G}_2\,\psi_2=\psi_1$, it suffices to verify the relation $\widehat{G}_2\,\xi^{(j)}{}_m=(-1)^{j-m}\xi^{(j)}{}_{-m}$. For example, $\mathrm{j}=1/2,\widehat{\mathrm{G}}_2\,\psi_1=\psi_2,,\widehat{\mathrm{G}}_2\psi_2=-\psi_1$.

Theorem 2

The spatial inversion $\widehat{\mathrm{P}}:\mathrm{t}'=\mathrm{t},\mathrm{x_i}'=\text{-}\mathrm{x_i}$ leads to an internal inversion $\widehat{\mathrm{I}}:\varphi'=\pi+\varphi,\theta'=\pi-\theta,\nu'=\pi-\nu$ – equivalent to the rotation in the isovector space by the angle $-\pi$ around the axis $\mathrm{X}^{(2)},\widehat{\mathrm{I}}=\exp(\mathrm{i}\,\pi\mathrm{J}^{(2)})$.

Proof

Let us prove the properties. We have:

$$\widehat{\mathrm{I}}\mathrm{X}^{(k)}=(\text{-}1)^{k}\,\mathrm{X}^{(k)},\ \widehat{\mathrm{I}}\xi^{(\pm)}=(\text{-}1)^{\mathrm{j}\mp\mathrm{j}}\,\xi^{(\mp)},\widehat{\mathrm{I}}\mathrm{J_i}=\mathrm{J_i},\widehat{\mathrm{I}}^2\xi^{(\pm)}=(\text{-}1)^{2\mathrm{j}}\,\xi^{(\pm)}.$$

The conservation law for parity is immediately related to the conservation for symmetry between left and right. The bases $\overline{D}^{j}{}_{m,m'}$ and $\overline{D}^{j}{}_{m,-m'}$ are related by the inversion transformation, $\widehat{I}\overline{D}^{j}{}_{m,m'}=(-1)^{j-m'}\overline{D}^{j}{}_{m,-m'}$. This enables one to decompose the entire basis into two equal groups of bases in all cases except for $m'=0$, the integer j. The group of bases $m'>0$ is said to be left, and its mirror part $m'<0$ to be right. The difference between left and right with weight m' is evaluated as the mean value of the operator $J^{(3)}$. For example, $\Psi=C^{(m')}\overline{D}^{j}{}_{m,m'}+C^{(-m')}\overline{D}^{j}{}_{m,-m'},<\Psi\left|J^{(3)}\right|\Psi>b^2=m'(\left|C^{(m')}\right|^2-\left|C^{(-m')}\right|^2)$.

Any translation of coordinates x_i, t does not change the angular variables φ,θ,ν .

THE GENERALIZED SPIN LORENTZ GROUP

The Lie algebra of the Lorentz group and its generators in the coordinate representation $\mathbf{L}=(\mathrm{M}_{23},\mathrm{M}_{31},\mathrm{M}_{12}),\mathbf{K}=(\mathrm{M}_{01},\ \mathrm{M}_{02},\ \mathrm{M}_{03})$ have this form (Ohnuki, 1988):

$$[L_i,L_n]=i\varepsilon_{ink}L_k,[K_i,L_n]=i\varepsilon_{ink}K_k,[K_i,K_n]=-i\varepsilon_{ink}L_k \quad (6)$$

Where $\mathrm{M_{ij}}=\mathrm{x_iP_j}-\mathrm{x_jP_i},\mathrm{x_0}=\mathrm{t},\mathrm{P_i}=-\mathrm{i}\partial/\partial\mathrm{x_i},\mathrm{P_0}=\mathrm{i}\partial/\partial\mathrm{t}$ stand for the operators of angular momentum, momentum and energy.

Let $\mathrm{Q}^{(\mathrm{p})}{}_{\mathrm{i}}$ be the generators of the Lorentz group for arbitrary spin in the angular representation. The Lorentz transformation is of the form $\psi'=\exp(-\mathrm{i}\chi\mathrm{Q}^{(\mathrm{p})}{}_{\mathrm{n}})\psi$, where $\mathrm{th}(\chi)=\mathrm{v}/c,\mathbf{v}=|\,\mathrm{v}|\,\mathbf{x}_n$ stands for the velocity, and p = 1, 2, 3, and

$$[J_i,J_n]=i\varepsilon_{ink}J_k,[Q^{(p)}{}_i,J_n]=i\varepsilon_{ink}Q^{(p)}{}_k,[Q^{(p)}{}_i,Q^{(p)}{}_n]=-i\varepsilon_{ink}J_k \quad (7)$$

Let us find the form of generators of Equation 7 satisfying the conditions that these are purely imaginary vector generators (which are Hermitean-first-order linear differential operators) and ensure the transformation properties of the basis Ψ, j =1/2,1,3/2… in accordance with the Lorentz transformations. To this end, we assume that the representation p = 3 has the Lorentz transformation on the basis $\psi=(\xi^{\mathrm{j})},\xi^{(-\mathrm{j})})$ or on the basis of eigen functions of $\mathrm{J_3},\mathrm{J}^{(3)},\mathbf{J}^2$ in the well-known diagonal form (Weinberg, 2003).

$$\mathrm{B}=\exp(-\mathrm{i}\chi\mathrm{Q}^{(3)}{}_3),\mathrm{B}\xi^{(\pm\mathrm{j})}{}_{\mathrm{m}}=\exp(\pm\mathrm{m}\chi)\xi^{(\pm\mathrm{j})}{}_{\mathrm{m}} \quad (8)$$

Therefore, the functions $\xi^{(\pm\mathrm{j})}{}_{\mathrm{m}}$ by themselves are eigen functions of the operator $\mathrm{Q}^{(3)}{}_3$. We impose the space-time isotropy condition on the generators of the Lorentz group, namely, $\mathrm{Q}^{(3)}{}_3$ must be independent of the angles φ and γ. Therefore, $\sin(\theta/2),\cos(\theta/2)$ are also eigen functions of the operator $\mathrm{Q}^{(3)}{}_3$. It can readily be shown that there is only one generator of this kind, $\mathrm{Q}^{(3)}{}_3=\mathrm{i}\,\mathrm{j}\cos(\theta)\text{-}\mathrm{i}\sin(\theta)\ \partial/\partial\theta,\quad \mathrm{Q}^{(3)}{}_3\xi^{(\pm\mathrm{j})}{}_{\mathrm{m}}=\pm\mathrm{i}\mathrm{m}\xi^{(\pm\mathrm{j})}{}_{\mathrm{m}}$.

Using $\mathrm{Q}^{(3)}{}_3$, we find $\mathbf{Q}^{(3)}=\mathrm{i}\mathbf{X}^{(3)}\mathrm{j}+[\mathbf{X}^{(3)}\mathbf{J}]$. Applying rotation in the isovector space, $\exp(\mathrm{i}\,\pi/2\,\mathrm{J}^{(2)})\,\mathrm{X}^{(3)}{}_{\mathrm{k}}=\mathrm{X}^{(1)}{}_{\mathrm{k}}$, $\exp(\mathrm{i}\,\pi/2\,\mathrm{J}^{(1)})\,\mathrm{X}^{(3)}{}_{\mathrm{k}}=\text{-}\,\mathrm{X}^{(2)}{}_{\mathrm{k}}$, we obtain two other generators p = 1, 2, the angular representations of the Lorentz group,

$$\mathbf{Q}^{(\mathrm{p})}=\mathrm{i}\mathbf{X}^{(\mathrm{p})}\mathrm{j}=+[\mathbf{X}^{(\mathrm{p})}\mathbf{J}],\mathrm{Q}^{(\mathrm{p})}{}_{\mathrm{k}}=\mathrm{i}\,\mathrm{X}^{(\mathrm{p})}{}_{\mathrm{k}}\ \mathrm{j}+\varepsilon_{\mathrm{ijk}}\,\mathrm{X}^{(\mathrm{p})}{}_{\mathrm{i}}\,\mathrm{J_j} \quad (9)$$

The dependence of the operators Q on j is excluded by replacing j with the scalar operator $\widehat{\mathrm{J}}$ independent of j and such that $\widehat{\mathrm{J}}\Psi=\mathrm{j}\Psi$.

Consider action of the operators J, Q for j = 1 on the basis $j=1,X^0{}_0=1$,

$$J^{(i)}X^{(n)}{}_p=-i\varepsilon_{ink}X^{(k)}{}_p,J_iX^{(p)}{}_n=i\varepsilon_{ink}X^{(p)}{}_k,Q^{(n)}{}_kX^0{}_0=ijX^{(n)}{}_k,$$
$$Q^{(m)}{}_iX^{(p)}{}_n=i\delta_{in}\delta_{mp}X^0{}_0+i\varepsilon_{ink}\varepsilon_{mpd}X^{(d)}{}_k \quad (10)$$

The operators Q, X, J and N with the superscripts and subscripts +,

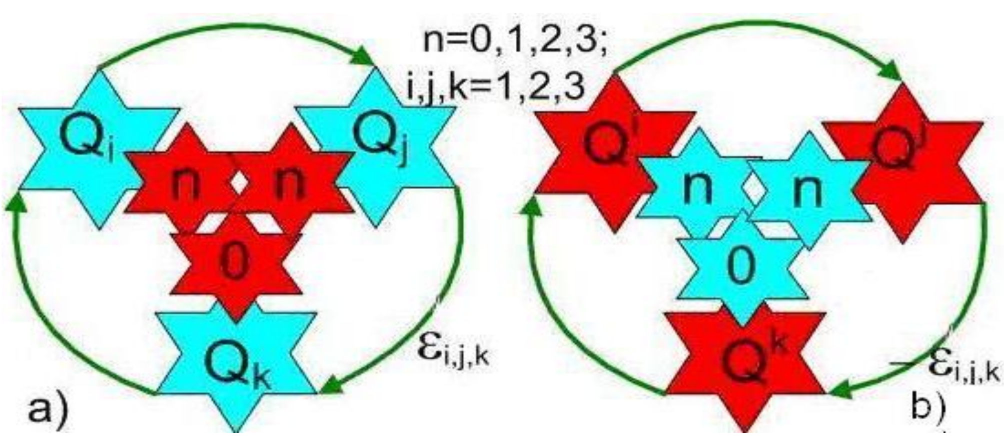

Figure 2. Visual model of Lie algebra and the generalized spin Lorentz group.

3, and −, correspondingly, raise, preserve, and lower, respectively, the indices m, m' for $\overline{D}^{j}{}_{m,m'}$. The result of action of the operators Q on $\overline{D}^{j}{}_{m,m'}$ (by using Varshalovich D.A, 1975) is just a linear combination of two summands $\overline{D}^{j}$ and $\overline{D}^{j-1}$. For example,

$$\begin{aligned} &Q^{(3)}{}_3\overline{D}^{j}{}_{m,m'} = i\sqrt{(j^2-m^2)(j^2-m'^2)}\,/\,j\overline{D}^{j-1}{}_{m,m'} + imm'/j\overline{D}^{j}{}_{m,m'}, \\ &Q^{(+)}{}_+\overline{D}^{j}{}_{m,m'} = i\sqrt{(j-m-1)(j-m)\,(j-m'-1)(j-m')}\,/\,j\overline{D}^{j-1}{}_{m+1,m'+1} + \\ &i\sqrt{(j-m)(j+m+1)}\sqrt{(j+m'+1)\,(j-m')}\,/\,j\overline{D}^{j}{}_{m+1,m'+1} \end{aligned} \tag{11}$$

The action of the operators $\mathbf{J},\mathbf{Q}^{(3)}$ on the basis $\psi=(\xi^{(j)},\xi^{(-j)}), S_{0ik}=j\delta_{ik}$ is of the form:

$$Q_k^{(3)}\psi = i\psi\begin{bmatrix} S_k & 0 \\ 0 & -S_k \end{bmatrix}, J_k\psi = \psi\begin{bmatrix} S_k & 0 \\ 0 & S_k \end{bmatrix}, J^{(3)}\psi = \psi\begin{bmatrix} S_0 & 0 \\ 0 & -S_0 \end{bmatrix}$$

Matrix representation of Equation (7) is identical to the spinor representation of the Lorentz group.

Similarly, we introduce the operators (Ohnuki, 1988) $\widehat{J}^{(1)}{}_i=(J_i+iQ^{(3)}{}_i)/2, \widehat{J}^{(2)}{}_n=(J_n-iQ^{(3)}{}_n)/2$ with the Lie algebra of the Lorentz group which coincides with the algebra of groups SO(3) and SU(2):

$$[\widehat{J}^{(1)}{}_i,\widehat{J}^{(2)}{}_n]=0,[\widehat{J}^{(1)}{}_i,\widehat{J}^{(1)}{}_j]=i\varepsilon_{ijk}\widehat{J}^{(1)}{}_k,[\widehat{J}^{(2)}{}_i,\widehat{J}^{(2)}{}_j]=i\varepsilon_{ijk}\widehat{J}^{(2)}{}_k\ .$$

It is known (Ohnuki, 1988:24) that an irreducible representation of the Lorentz group is uniquely determined by eigen values of some operators $\mathbf{L}^2-\mathbf{K}^2,\mathbf{LK}$, which correspond to two operators $\widehat{\mathbf{J}}^{(1)2},\widehat{\mathbf{J}}^{(2)2}$. So $\widehat{\mathbf{J}}^{(1)2}\Psi=j_1(j_1+1),\widehat{\mathbf{J}}^{(2)2}\Psi=j_2(j_2+1)$. Every irreducible representation of the Lie algebra is characterized by a pair of numbers (j_1,j_2).

The spinors $\xi^{(j)}$ and $\xi^{(-j)}$ are transformed accordingly to the representations of (0, j) and (j, 0), respectively (0,1/2), (1/2,0), which are the Weyl spinors.

Scalar unit operator 1 and the generators of the groups ensuring the vector, isovector and Lorentz rotations of the bases in the new variables generate an generalized spin Lorentz group of 16 generators, Equations (4) and (12).

$$\begin{aligned} &[\mathbf{Q}^{(n)}{}_i,\mathbf{Q}^{(p)}{}_k]=i\delta_{ik}\varepsilon_{npm}\mathbf{J}^{(m)}-i\delta_{np}\varepsilon_{ikr}\mathbf{J}_r, \\ &[\mathbf{Q}^{(i)}{}_r,\mathbf{J}^{(n)}]=-i\varepsilon_{ink}\mathbf{Q}^{(k)}{}_r,[\mathbf{Q}^{(p)}{}_i,\mathbf{J}_n]=i\varepsilon_{ink}\mathbf{Q}^{(p)}{}_k \end{aligned} \tag{12}$$

Visual model of Lie algebra and the generalized spin Lorentz group is shown in Figure 2.

In Figure 2, $J^{(k)}$ corresponds to $Q^{(k)}{}_0$; J_n corresponds to $Q^{(0)}{}_n$. The operator $W:\varphi'=\nu,\nu'=\varphi,\theta'=-\theta$ of permutation introduces the ordinary observer and the local observer or the vector and isovector rotations. The operation W preserves the invariance of the Lie algebra of the generalized spin Lorentz group and corresponds to the transposition operation for the group representation and matrix $\overline{D}^{j}$.

$$J^{(k)}=(-1)^{k+1}W\,J_k, W\overline{D}^{j}=\overline{D}^{jT}, X^{(k)}{}_n=(-1)^{k+n}W\,X^{(n)}{}_k\,,\ Q^{(k)}{}_n=(-1)^{k+n}W\,Q^{(n)}{}_k\,.$$

The ten generators of the Poincaré group $(\mathbf{L},\mathbf{K},\mathbf{P},P_0)$ and the ten operators $(\mathbf{J},\mathbf{Q}^{(3)},\mathbf{Q}^{(-)},iJ^{(-)})$ have the same Lie algebra in Equations 6 and 13; they are preserved under the cyclic permutation of the superscripts 1, 2, 3.

$$[P_i,P_n]=0,[K_k,P_k]=-iP_0,[L_i,P_j]=i\varepsilon_{ijk}P_k,[K_i,P_0]=-iP_i\,, \tag{13}$$

The rotation of the coordinate system X_1,X_2,X_3,t by the angle ζ_k around the x_k axis and the Lorentz transformation corresponding to the velocity v = |v|x_n are of the form

$$\Psi'=\exp(-i\zeta_k(L_k+J_k))\Psi, \Psi'=\exp(-i\chi\ (K_n+Q^{(3)}{}_n))\Psi \tag{14}$$

The extended basis $X^{(0)}{}_0-\mathbf{X}^{(3)}$ is transformed as a four-dimensional vector, the basis $(-\mathbf{X}^{(1)},\mathbf{X}^{(2)})$ is transformed as a Lorentz bivector, and all these objects together form a full system of 10 bases for j = 1. The basis $X^{(0)}{}_0=\psi_1\psi_2-\psi_3\psi_4=1$ differs from the first basis for j = 0 only in the dimension and Lorentz transformation

$$\begin{aligned} &BX^{(3)}{}_3=ch(\chi)X^{(3)}{}_3+sh(\chi)X^{(0)}{}_0, BX^{(0)}{}_0=sh(\chi)X^{(3)}{}_3+ch(\chi)X^{(0)}{}_0, \\ &X^{(+)'}{}_+=X^{(+)}{}_+\exp(\chi), X^{(+)'}{}_-=X^{(+)}{}_-\exp(-\chi), \end{aligned} \tag{15}$$

Transformation operators are performed according to the Baker-

Campbell-Hausdorff (Biedenharn and Louck, 1984): $\exp(A)B\exp(-A)=B+[AB]+1/2![A[AB]]+1/3![A[A[AB]]]+..$

The operators $\mathbf{Q}^{(2)},\mathbf{J}^{(1)}$ and $\mathbf{Q}^{(1)},-\mathbf{J}^{(2)}$ or $\mathbf{Q}^{(-)},i\,\mathbf{J}^{(-)}$ are transformed as a four-dimensional Lorentz vector, $\mathbf{Q}^{(3)},\mathbf{J}$ as a bivector, and $J^{(3)}$ as a scalar,

$$Q^{(2)}{}_1{}' =' Q^{(2)}{}_1, Q^{(2)}{}_2{}' = Q^{(2)}{}_2, \mathbf{v}/v=\mathbf{x}_3, Q^{(3)}{}_3{}' = Q^{(3)}{}_3, J_3' = J_3,$$

$$\begin{bmatrix} J_1' \\ Q^{(3)}{}_2{}' \end{bmatrix} = \begin{bmatrix} ch(\chi) & sh(\chi) \\ sh(\chi) & ch(\chi) \end{bmatrix} \circ \begin{bmatrix} J_1 \\ Q^{(3)}{}_2 \end{bmatrix}$$

Proposition 3

The Lorentz transformation of the basis $\mathrm{j}, \mathbf{v}=|\mathrm{v}|\mathbf{x_3}$ is equivalent to an angular transformation and a scale transformation of the basis.

$$\varphi'=\varphi, \nu'=\nu, tg(\theta'/2)=\exp(-\chi)tg(\theta/2), \psi'_{eff}=\psi/T(\chi,\theta)^{2j} \quad (16)$$

Where $T(\chi,\theta)^2=ch(\chi)+sh(\chi)\cos(\theta)$.

The operator B in the above Lorentz transformation does not depend on the angles φ, γ. The Lorentz transformation of the angles φ,θ,γ follows from the Lorentz transformation of Equations 8 and 15 of the bases $\xi^{(\pm j)}{}_m, X^{(k)}{}_i$. The Lorentz transformation of Equation 15 of $\mathbf{X}^{(1)},\mathbf{X}^{(2)},\mathbf{X}^{(3)}$ is dual to the Lorentz transformation correspondingly, for the vector E of the electric field, the vector H of the magnetic field, and the Umov-Poynting vector [EH], (Pauli W., 1991). Therefore, there exists an analogy between the Lorentz transformation of Equation 16 of the angles φ,θ,ν and the Lorentz transformation of the angles of the polarized light, where the Umov-Poynting vector is directed along the vector $\mathbf{X}^{(3))}$ (Figure 1) and the electric vector along the vector $\mathbf{X}^{(1)}$.

The independence of the speed of light of the reference system implies that the shape of spin states for a massless particle with spin j that flies along the x_3 is invariant under the Lorentz transformation along x_3. The only eigen functions of the generator of the Lorentz group $Q^{(3)}{}_3, \mathbf{J}^{(2)}, \Psi=\xi^{(\pm)}{}_m, \Psi=\xi^{(\pm j)}{}_m, \Psi=\xi^{(m)}{}_{\pm j}$ can satisfy this invariance condition. The spinors $\xi^{(\pm j)}{}_m$ can be treated as a basis of the massless field. For j =1/2 let us show that the neutrinos are purely left handed particles $<\Psi|J^{(3)}|\ \Psi>b^2=1/2$, whereas the antineutrinos are the right-handed ones $<\Psi|J^{(3)}|\ \Psi>b^2=-1/2$. To this end, we choose the increasing solutions under the Lorentz transformation of Equation 8 along the x_3 axis; these are $\Psi=\xi^{(1/2)}{}_{-1/2}$ for the neutrino and $\Psi=\xi^{(-1/2)}{}_{1/2}$ for the antineutrino.

The vector $\mathbf{P}^{(-)},\mathrm{P}^{(-)}{}_0$ can be considered as the momentum operator in the Lie algebra of the angular variables, as they have the same Lie algebra of the Poincaré group, so that

$$\mathbf{P}^{(-)}=\mathbf{Q}^{(-)}, P^{(-)}{}_0=i\,J^{(-)}, P^{(1)}=Q^{(1)}, P^{(2)}=Q^{(2)}, P^{(1)}{}_0=-J^{(2)}, P^{(2)}{}_0=J^{(1)}, \mathbf{P}^{(-)}=\mathbf{P}^{(1)}+i\mathbf{P}^{(2)}.$$

We introduce the operators in the symmetric form of the Lorentz invariant scalar product of two four-vector momentum operators **P**, P_0/c and $\mathbf{P}^{(-)}$, $P^{(-)}{}_0$ or $P^{(+)}$, $P^{(+)}{}_0$.

In the equation for the wave function with arbitrary spin, we will use this operator $N^{(\pm)}=\mathbf{P}^{(\pm)}\mathbf{P}-P^{(\pm)}{}_0$, where $N^{(-)}=N^{(1)}+iN^{(2)}$ is the complexified operator.

Below we will use the invariant real-valued operators

$$N^{(1)}=\mathbf{Q}^{(1)}\mathbf{P}+J^{(2)}P_0/c, N^{(2)}=\mathbf{Q}^{(2)}\mathbf{P}-J^{(1)}P_0/c, \quad (17)$$

instead of the Lorentz invariant operators $N^{(\pm)}$. These operators have different parity $\widehat{P}\widehat{I}N^{(1)}=N^{(1)}, \widehat{P}\widehat{I}N^{(2)}=-N^{(2)}, \widehat{I}\mathbf{Q}^{(2)}=\mathbf{Q}^{(2)}$ and are connected to each other by rotations around a movable axis $X^{(3)}$ in angles $\pi/2, \nu'=\nu-\pi/2, N^{(1)}=\exp(i\,\pi/2\,J^{(3)})N^{(2)}$

THE GENERALIZED DIRAC EQUATION

Consider the generalized Dirac equation for a particle with arbitrary spin for для $j=1/2,1,3/2...,..$ written either in terms of $N^{(1)}$ or in terms of $N^{(2)}$, k = 1 or k = 2:

$$N^{(k)}\Psi=m_e c/(2h)\Psi \quad (18)$$

This is the first-order linear differential equation for wave functions $\Psi=b\psi C$. Consider the set of the eigenvectors of the operators $J_3,\mathbf{J}^2,J^{(3)}$, consisting of (2j +1)(2j +1) of functions and let me be the mass of the particle and h be the Planck's constant. Equation 11 readily shows that the generalized Dirac equation reduces the staircase equations for the $\Psi^{(j)},\Psi^{(j-1)}$.... This explains the fact that if j > 1/2 then particles or fields are composite. For example, for j = 3/2, presence of the physical modes is obligatory for the Rarita - Schwinger fields with spin 1/2.

The matrix representation of equations (18) on the basis (5) with j = 1/2, k = 2 coincides with the following Dirac equations in spinor representation (Fushchich and Nikitin, 1994)

$$(gP-g_0P_0/c+m_ec/h)=0 \quad (19)$$

Where

$$g_0=\begin{bmatrix}0 & \sigma_0\\ \sigma_0 & 0\end{bmatrix}, g_k=\begin{bmatrix}0 & -\sigma_k\\ \sigma_k & 0\end{bmatrix}, \widehat{g}_0=i\begin{bmatrix}0 & \sigma_0\\ -\sigma_0 & 0\end{bmatrix}, \widehat{g}_k=-i\begin{bmatrix}0 & \sigma_k\\ \sigma_k & 0\end{bmatrix},$$

σ are the Pauli matrices, g_0, g and $\widehat{g}_0$, $\widehat{g}$ are the Dirac matrices in two different spinor representations. To prove Equation 19, it suffices to calculate the action of the operators $N^{(1)}$,$N^{(2)}$ on the basis vectors of Equation 5, $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)$ for j = 1/2.

$$2N^{(2)}\psi=\psi(\mathbf{gP}-g_0P_0/c), 2N^{(1)}\psi=\psi(\widehat{\mathbf{g}}\mathbf{P}-\widehat{g}_0P_0/c).$$

In order to simplify the above equations we have used the following identities:

$$2J^{(1)}\psi=\psi g_0, 2Q^{(2)}\psi=\psi g_k, -2Q^{(1)}\psi=\psi\widehat{g}_k, 2J^{(2)}\psi=\psi\widehat{g}_0$$

$$g_0=<2\psi|J^{(1)}|\ \psi>b^2, g_k=<2\psi|Q^{(2)}|\ \psi>b^2,$$

$$2J_k\psi=\psi\begin{bmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{bmatrix}, 2J^{(3)}\ \psi=\psi\begin{bmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{bmatrix}, <\psi|\ \psi>b^2=\begin{bmatrix}\sigma_0 & 0\\ 0 & \sigma_0\end{bmatrix}$$

The generalized Dirac equation is invariant under inversion of $\widehat{\mathrm{P}}$, $\widehat{\mathrm{I}}$ and the solution $\Psi = \Psi^C$ for a charge-conjugate particle satisfies the complex conjugate Equation 18:

$$\overline{\mathrm{Q}}^{(n)})_k = -\mathrm{Q}^{(n)})_k, (\overline{\mathrm{P}} - q\overline{\mathrm{A}}/c) = -(\mathrm{P} + q\mathrm{A}/c), \mathrm{J}^{(k)} = -\mathrm{J}^{(k)}, \Psi^C = (\overline{\psi}\overline{\mathrm{C}}) = (\psi \mathrm{C}^C),$$

Where A is the vector-potential, q/h is the charge of a particle. For example, if j = 1/2, then $\overline{\psi} = i\psi g_2, C^C = ig_2 C$, (Flügge, 1974).

For spin j = ½, there exists 2 = 2j + 1 Lorentz-invariant states $\Psi^{(\pm)} = C^{(\pm)}\xi^{(\pm 1.2)}$. The generalized Weyl equations for the right- or left-handed neutrino have the form $\mathrm{N}^{(\pm)}\Psi^{(\mp)} = 0$ or $(-\mathrm{P}_0 c \pm \mathbf{P\sigma}) C^{\pm} = 0$, (Akhiezer, 1959). For the anti neutrino and neutrino, the balance between left and right orientations is violated because of asymmetry of equations $\mathrm{J}^{(3)}\Psi^{(\pm)} = \pm\Psi^{(\pm)}/2$.

THE GENERALIZED MAXWELL EQUATIONS

The Maxwell equations describing the state of the vector field E, H, have the following form:

$$\begin{aligned}[\mathbf{PE}] &= \mathrm{P}_0\mathbf{H}/c, (\mathbf{PH}) = 0,\\ [\mathbf{PH}] &= -\mathrm{P}_0\mathbf{E}/c - i4\pi/c\mathbf{I}, (\mathbf{PE}) = -i4\pi/c\mathrm{I}_0\end{aligned} \tag{20}$$

Where $\mathbf{I} = (I_1, I_2, I_3)$ is the density of electric current and I_0 is the density of electric charge. Let

$$\mathbf{F}^{(1)} = \mathbf{E}, \mathbf{F}^{(2)} = \mathbf{H}, \mathbf{F}^{3)} = \mathbf{A}, \mathrm{b} = \sqrt{(2j+1)}/(4\pi).$$

Consider the generalized Maxwell's equations which imply the standard Maxwell equations for E and H which are not just a new representation, but a tool for an expanded description of the states of electromagnetic fields with spin. To this end, we introduced the wave function Ψ. The Lorentz-invariant of the wave function Ψ_C for charges and currents has the following form:

$$\Psi_C = (\mathrm{I}_0\,\mathrm{X}^{(0)}{}_0 + \mathrm{I}\,\mathrm{X}^{(3)}/c), \mathrm{I}_0 = 1/(4\pi)^2 < \Psi_C|\mathrm{X}^{(3)}{}_k>, \mathrm{I}_k = \mathrm{b}^2 < \Psi_C|\mathrm{X}^{(3)}{}_k>.$$

Let us describe the state Ψ on the basis of eigen functions of $\mathrm{J}_3, \mathbf{J}^2, \mathrm{J}^{(3)}, j = 1, \mathbf{J}^2\Psi = 2\Psi$ and consider all three cases $\mathrm{J}^{(3)}\Psi = \pm\Psi, \mathrm{J}^{(3)}\Psi = 0$.

This Ψ is composed of 3=2j +1, (j=1) Lorentz-invariants of the electromagnetic field written in the form of the two complex-conjugate invariants $\Psi^{(\pm)} = \mathrm{C}^{(\pm)}\xi^{(\pm 1)} = \mathbf{F}^{(\mp)}\mathbf{X}^{(\pm)}$, the two real invariants of $\Psi = \mathbf{E}\,\mathbf{X}^{(1)} + \mathbf{H}\,\mathbf{X}^{(2)}, \Psi_{AN} = \mathbf{E}\,\mathbf{X}^{(2)} - \mathbf{H}\,\mathbf{X}^{(1)}$ and the invariant $\Psi_A = \mathbf{A}\,\mathbf{X}^{(3)} + \mathrm{A}_0\mathrm{X}^{(0)}{}_0$ for the vector-potential $\mathbf{A}, \mathrm{A}_0, J^{(3)}\Psi_A = 0$. The states $J^{(3)}\mathbf{\Psi}^{(\pm)} = \pm\mathbf{\Psi}^{(\pm)}$ are called the right and the left vector states, respectively.

The Lorentz-invariance of the $\Psi^{(+)}$ implies the well-known Lorentz transformations of the fields (Pauli, 1991): $\mathrm{F}^{(-)}{}'_{-} = \mathrm{F}^{(-)}{}_{-}\exp(-\chi), \mathrm{F}^{(-)}{}'_3 = \mathrm{F}^{(-)}{}_3, \mathrm{F}^{(-)}{}'_{+} = \mathrm{F}^{(-)}{}_{+}\exp(\chi)$.

For the state $\Psi^{(+)}$, we have

$$\mathbf{F}^{(-)}\mathbf{X}^{(+)} = (\mathrm{F}^{(-)}{}_{+}\,\mathrm{X}^{(+)}{}_{-} + 2\mathrm{F}^{(-)}{}_3\,\mathrm{X}^{(+)}{}_3 + \mathrm{F}^{(-)}{}_{-}\,\mathrm{X}^{(+)}{}_{-})/2 = C_R\xi^{(1)},$$

$\overline{D}^1{}_{\pm1,1} = \pm X^{(+)}{}_{\pm}, \overline{D}^1{}_{0,1} = -X^{(+)}{}_3/\sqrt{2}$. The basis $\xi^{(1)} = (\overline{\mathrm{D}}^1{}_{1,1}, \overline{\mathrm{D}}^1{}_{0,1}, \overline{\mathrm{D}}^1{}_{-1,1})$ and the amplitudes $C^{(+)} = (\mathrm{F}^{(-)}{}_{+}, 2\,\mathrm{F}^{(-)}{}_3, \mathrm{F}^{(-)}{}_{-})$ correspond to the projections of the spin (1, 0,−1) on the x_3 axis. We expand similarly $\Psi = \Psi^{(-)}$.

The spins of the right and left vector states are collinear or anticollinear to the Poynting vector. The proof follows from the identities $\mathrm{b}^2 < \Psi^{(\pm)}|\mathbf{J}|\,\Psi^{(\pm)}> = \pm 4[\mathbf{EH}]$.

The Poynting vector and the density of the electromagnetic energy $\mathbf{s} = c[\mathbf{EH}]/(4\pi), s_4 = (\mathrm{E}^2 + \mathrm{H}^2)/(8\pi)$ for $\Psi = \Psi^{(\pm)}$ are written in the form:

$$2\mathbf{s} = -ic\mathrm{b}^2 < \Psi|\mathbf{Q}^{(3)}|\,\Psi>/(8\pi), 2s_4 = \mathrm{b}^2 < 2\Psi|\,\Psi>/(8\pi).$$

The mixed state is the sum of the right and the left vector states, currents and fields:

$$\Psi = \Psi^{(+)} + \Psi^{(-)}, \mathrm{E} = \mathrm{E}_R + \mathrm{E}_L, \mathrm{H} = \mathrm{H}_R + \mathrm{H}_L, \mathrm{I} = \mathrm{I}_R + \mathrm{I}_L, \mathrm{I}_0 = \mathrm{I}_{0R} + \mathrm{I}_{0L}, \Psi_C = \Psi_{CR} + \Psi_{CL},$$

Proposition 4

The spin (the energy, the Poynting vector) of the mixed state is the sum of the spins (the energy, the Poynting vector) of the right and the left states.

Proof follows from the identity, similarly for $Q^{(3)}$: $<\Psi|\mathbf{J}|\,\Psi> = <\Psi^{(+)}|\mathbf{J}|\,\Psi^{(+)}> + <\Psi^{(-)}|\mathbf{J}|\,\Psi^{(-)}>$. The generalized Maxwell equations describing just right state or only left state with spin 1 have the following form:

$$\mathrm{N}^{(1)}\mathbf{\Psi}^{(\pm)} = 4\pi\mathbf{\Psi}^c, \text{or}, \mathrm{N}^{(2)}\mathbf{\Psi}^{(\pm)} = \mp i4\pi\mathbf{\Psi}^c \tag{21}$$

We rewrite Equations 21 as a pair of complex-conjugate Equations 22 for the wave functions either $\Psi^{(+)}$ or $\Psi^{(-)}$, as $N^{(\pm)}\Psi^{(\pm)} = 0$. Each of these equations describes just the right or the left vector spin-1 states:

$$\mathrm{N}^{(-)}\Psi^{(+)} = 8\pi\Psi^c, \mathrm{N}^{+)}\Psi^{-)} = 8\pi\Psi^c, \tag{22}$$

Each of the Equations 22 agrees with the Maxwell equations either for the right components E_R, H_R, I_R, I_{0R} or for the left components. This implies that the total components $\mathrm{E}, \mathrm{H}, \mathrm{I}, \mathrm{I}_0$ satisfy the Maxwell equations. Maxwell Equation 20 follow from Equations 21, 10 and identity 23 for j = 1, where $G(x_1, x_2, x_3)$ is an arbitrary vector:

$$(\mathbf{Q}^{(i)}\mathbf{P})(\mathbf{G}\mathbf{X}^{(k)}) = i\,\delta_{ik}\,(\mathbf{PG})\mathrm{X}^{(0)}{}_0 + i\,\varepsilon_{ikn}\,([\mathbf{PG}]\mathrm{X}^{(n)}). \tag{23}$$

For $\Psi^{(+)} = \mathbf{F}^{(-)}\mathbf{X}^{(+)}, \mathbf{G} = \mathbf{F}^{(-)}$ we obtain the following identity:

$$(|\mathrm{Q}^{(2)}\mathrm{P} - i\mathrm{J}^{(1)}\mathrm{P}_0/c)(\mathbf{F}^{(-)}\mathbf{X}^{(+)}) = -i([\mathbf{PF}^{(-)}] + i\mathrm{P}_0\mathbf{F}^{(-)}/c)\mathbf{X}^{(3)} + (\mathbf{PF}^{(-)})\mathrm{X}^{(0)}{}_0$$

The generalized Maxwell Equation 24 for states with zero projection of spin and its analogue $<\Psi|\mathrm{J}_i|\,\Psi> = 0, <\Psi|\mathrm{J}^{(k)}|\,\Psi> = 0$ correspond to the real-valued state $\Psi = \mathbf{E}\,\mathbf{X}^{(1)} + \mathbf{H}\,\mathbf{X}^{(2)}$. Since $\Psi, N^{(2)}$ is the real one, then $C_L = C_R$ and Equations 22 have the following form:

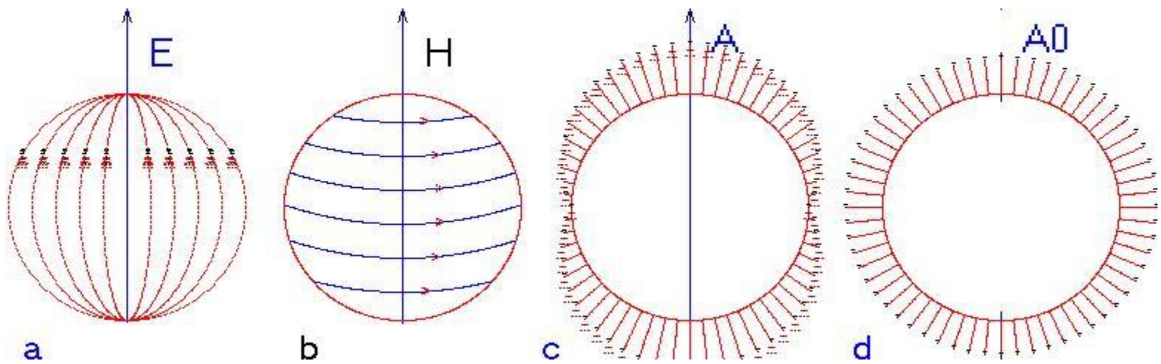

Figure 3. Visual model of the electric, magnetic fields: $\sin(\theta)\mathbf{i}_\theta$, $\sin(\theta)\mathbf{i}_\varphi$, $\cos(\theta)\mathbf{i}_r$, $\mathbf{i}_r$

$$\mathrm{N}^{(1)}\Psi = 4\pi\Psi^c, \mathrm{N}^{(2)}\Psi = 0 \; . \qquad (24)$$

The transformation properties of Equation 15 of the basis $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$ and the **E, H** of electric and magnetic fields are dual to each other, so that $\mathbf{E} = \mathrm{b}^2 < \Psi|\mathbf{X}^{(1)} >, \mathbf{H} = \mathrm{b}^2 < \Psi|\mathbf{X}^{(2)} >$.

The matrix representation of Equations 24 on the basis $\mathrm{X}^{(k)}{}_i$ k coincides with the Maxwell Equation 20. Model of the electric field, magnetic field and the field of magnetic vector potential is represented in the form $\mathrm{X}^{(1)}{}_3, \mathrm{X}^{(2)}{}_3, \mathrm{X}^{(3)}{}_3, \mathrm{X}^{(0)}{}_0$; it is composed from D-Wigner functions for spin j = 1, after averaging over the angle γ (Figure 3). Vector is the set of its projections.

Theorem 3

The generalized Dirac equation $\mathrm{N}^{(1)}\Psi = -m_e c/(2h)\ \Psi$ for a particle with mass m_e, spin j = 1, and zero projection of the spin on any axis X_i and its analogue on any axis $X^{(k)}$ for the amplitudes on the basis $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \mathrm{X}^{(0)}{}_0$ are equivalent to the Maxwell and Proca equations, respectively, for the 1-spin particle of mass m_e (heavy vector virtual photon).

Proof

We represent the solution as a sum of Lorentz invariants

$$\Psi = \Psi_{\mathrm{EH}} + \Psi_{\mathrm{A}}/\Lambda, \Psi_{\mathrm{EH}} = \mathbf{E}\,\mathbf{X}^{(1)} + \mathbf{H}\,\mathbf{X}^{(2)}, \Psi_{\mathrm{A}} == \mathbf{A}\,\mathbf{X}^{(3)} + \mathrm{A}_0\,\mathrm{X}^{(0)}{}_0,$$

Where $\Lambda = (2h)/(m_e c)$.

Zero values of the spin projection and its counterpart are equivalent to the real values of the amplitudes of E,H, A,A_0. The generalized Dirac equation splits into two equations:

$$N^{(1)}\Psi_A = -\Psi_{EH}, or, \mathbf{H} = i[\mathbf{PA}], \mathbf{E} = iP_0\mathbf{A}/c - i\mathbf{P}A_0,$$
$$N^{(1)}\Psi_{EH} = -\Psi_A/\Lambda^2, or, i(\mathbf{PE}) + A_0/\Lambda^2 = 0, iP_0\mathbf{E}/c + i[\mathbf{PH}] + \mathbf{A}/\Lambda^2 = 0$$

The Maxwell equations imply the continuity equation for I_s which coincides with the Lorentz calibration of the vector potential of magnetic field, and the first London equation, which describes the Meissner effect of Equation 25, (London and London, 1935), where Λ is the London penetration depth, **A** is the vector potential, I_s is the superconducting component of the electric current.

$$\mathbf{A} = -\mathbf{I}_s 4\pi\Lambda^2/c, \quad A_0 = -4\pi\Lambda^2 I_{0s}, \quad [\mathbf{P}\,\mathbf{I}_s] = i\,\mathbf{H}c/(4\pi\Lambda^2) \qquad (25)$$

The quaternion forms of these equations are equivalent to the equation $(\mathbf{A}, A_0) = -4\Lambda^2(\mathbf{I}_s/c, I_{0s})$. Using the equation $\Diamond(\mathbf{A}, A_0) = 4\pi^2(\mathbf{I}/c, I_{0s}), \mathbf{I}_s = \mathbf{I}$, we obtain the equation $((\Diamond + 1/\Lambda^2)(\mathbf{A}, A_0) = 0$, written as the Proca equation (Ginzburg, 1979), where $\Diamond$ is the d'Alembert operator. The above equations can describe the superconductivity phenomena because the acquisition of mass by photons is associated with the losses of long-range interactions. This can reduce the energy losses by radiation.

RESULTS AND DISCUSSION

The reasons for the above proposed generalizations are related to the fact that fields E, H are not sufficient enough to describe the electromagnetic fields with spin; this is because the spin part is closely related to the wave function Ψ. By virtue of this, we introduce the wave function $\Psi = \Psi^{(+)} + \Psi^{(-)}$ which is equal to the sum of the left and right vector states with spin 1. The contribution of the left and right states $\Psi^{(+)}$ and $\Psi^{(-)}$ to the wave function is independent of the Maxwell equations. The spin of electromagnetic field E, H, Ψ is equal to the sum of spins of the left and the right states of the field. Any electromagnetic field E,H related to a stationary state can possess a spin. The generalized Maxwell equation admits spin states, but does not describe completely their variation. The spin of electromagnetic fields E,H corresponds to either the right or left of the vector. Spins of the left and right vector states E,H are collinear and anticollinear to the Poynting vector.

We propose the following mechanism for the transition of a conductor to superconducting state. The low-temperature superconductivity corresponds to a spontaneous transition to the state with zero spin projection and its analogues $< \Psi|J_i|\ \Psi >= 0, < \Psi|J^{(k)}|\ \Psi >= 0$, without changing the electromagnetic field E,H.

The high-temperature superconductivity can be considered as a transition to the state with spin 1. Besides, state Ψ must possess an analogue of spin 1, whose projection to $X^{(3)}$ axis is equal to $\pm 1, J^{(3)}\Psi = \pm\Psi$.

Our conjecture is the following: there exists just three generations of leptons (electrons, muons, tau-leptons), because the generalized spin group possesses at the same time just the three different spinor representations of the Lorentz group of Equation 7. The three representations of the Poincaré group derived by cyclic permutations of superscripts p = 1,2,3 correspond to the three representations of the Dirac equation. From a mathematical point of view, they are equivalent. The spinor representation of Poincaré group p = 3 associated with the first (stable in the decay) generation of leptons (electron) as well as the generator of the Lorentz transformation $Q^{(3)}{}_3$ is independent of the angles φ,γ.

We conjecture that there exists just three colors of quarks (red, green, blue), since the generalized spin Lorentz group possesses at the same time just the three different transposed spinor representations of the Lorentz group.

Minkowski space (non-locally isotropic in the presence of particles and fields) has three additional independent dimensions φ,θ,ν, which fully describe the degree of freedom of the spin.

Note that physical systems normally can be represented precisely in terms of purely complex generalized spin Lorentz groups, but not just by the Lorentz group in the spinor representation, due to the existence of an additional degree of freedom. Generalized spin Lorentz groups consist of the three Lorentz groups in spinor representations, and the three transposed Lorentz groups in spinor representations.

Similarly, spin of the particle, electromagnetic field (in state E, H, A) can be regarded as a consequence of the presence of internal degree of freedom (analog of spin 1 for isovector space) of Minkowski space.

Similarly (6,7), replacement of $\mathbf{L},\mathbf{K}^{(3)}$ to $\mathbf{J},\mathbf{Q}^{(3)}$,we introduce analog Pauli-Lubanski Spin Operator: $\widehat{W}_0 = (\mathbf{P}\,\mathbf{J}), \widehat{\mathbf{W}} = P_0\,\mathbf{J} - [\mathbf{P}\mathbf{Q}^{(3)}]$.

Conclusion

In this paper, spinors and their transformation properties are described as the properties of the rotation of local observer, but not in terms of rotation of the ordinary (meter) observer. This permutation of the local and ordinary observers agrees with the general theory of relativity. To this end, we introduce one-component wave functions $\Psi(x_1, x_2, x_3, t, \varphi, \theta, \nu)$, depending on the position and the orientation of a local observer in the Minkovski space-time determined by the three Euler angles φ,θ,ν. The seven variables $x_1, x_2, x_3, t, \varphi, \theta, \nu$ are independent, but their transformation properties are bound between them. For an arbitrary integer or half-integer spin j, the Poincaré group in angular (spinor) representation is described explicitly.

The matrix representation of the algebra $(\mathbf{J}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(-)}, iJ^{(-)})$ coincides with the Poincaré group in spinor representation. A matrix representation of 16 generators of the generalized spin Lorentz group on the basis of j = 1/2 coincides with 16 basic elements of the Clifford algebra formed by the Dirac gamma-matrices (Flügge, 1974). Momentum operator $\mathbf{Q}^{(-)}, iJ^{(-)}$ in the Lie algebra of the angular variables of the Poincaré group is always a complex operator. Analogs of discrete operator $\widehat{P}$ (reverse the orientation of space) are included in the generalized spin Lorentz group as continuous operator $\widehat{I} = \exp(i\pi J^{(2)}), \widehat{I}Q^{(3)} = -Q^{(3)}$. Discrete operator $\widehat{P}\widehat{T}$ ($\widehat{T}$ - reverse the time) corresponds to the operator of complex conjugation for P, P_0, Ψ. This corresponds to continuous operator $\bar{\xi}^{(\pm j)} = -\exp(-i\,\pi J^{(2)} - i\,\pi J_2)\xi^{(\pm j)}$.

The Lorentz group O(1, 3) has four connected components (Ohnuki, 1988). The elements in each component are characterized by whether or not they reverse the orientation of space and/or time. Generalized spin Lorentz group and the Lorentz group O(1, 3) have four connected components.

Also are derived in uniform manner the generalized Dirac and Maxwell equations for Ψ with arbitrary spin j in terms of scalar product of two four-vector momentum operators (generators the Poincaré group) in the angular representation and coordinate representation. The matrix representations of the corresponding operator equations for spin j = 1/2, 1 particles coincide with the Dirac, Maxwell, or Proca equations.

These generalized Dirac and Maxwell equations imply the following conclusions:

(1) Free fixed electron and positron have analogues of the spin projection ±1/2, $J^{(1)}\Psi = \pm\Psi/2$ along $\mathbf{X}^{(1)}$ axis because the term, depending on the momentum P in (18), k=2, vanishes. In the standard representation (Weinberg, 2003), this determines our choice of the basis: $\eta = \xi^{(1/2)} \pm \xi^{(-1/2)}$;

(2) Neutrinos and antineutrinos, left and right photons moving along x_3 are considered as states with spin projections on the axis x_3; analogues of the spin projections on the axis of $X^{(3)}$ are equal to ±1/2 and ±1, respectively. For neutrinos and antineutrinos, we have $j = 1/2,\ J^{(3)}\Psi = \pm\Psi/2, J_3\Psi = \mp\Psi/2$, where $\Psi = \psi_2, \Psi = \psi_3$; for photons $j = 1,\ J^{(3)}\Psi = \pm\Psi, J_3\Psi = \mp\Psi$, where $\Psi = X^{(\pm)}{}_{\mp}$. The Lorentz transformation along x_3 does not change form Ψ for neutrino, antineutrino and photons, since Ψ is an eigen function of generator Lorentz transformations $Q^{(3)}{}_3$.

The sum of the left and right photons $X^{(+)}{}_{-} + X^{(-)}{}_{+} = 2(X^{(1)}{}_{1} - X^{(2)}{}_{2})$ corresponds to the plane polarized photon;

(3) The generalized Dirac equation for a massive particle with j =1, zero spin projection and its analogue $J^{(k)}$, and the assumptions on implementation of the Maxwell equation imply the first London's equation for superconductivity which produces the Meissner effect and the possibility of creation of massive photons.

Using the known representation for the transposed spinor representation of rotation group (4) and spinor representation (7) of the Lorentz group of angular variables, which are unique for p = 3, we obtain the generalized spin Lorentz spin group (4),(12) corresponding to the unification of the spinor representation of the Lorentz group and its transposed representation. Unambiguous form of generalized spin Lorentz group follows from the principle of the special relativity theory on equivalence of coordinate systems with respect to permutation of local observer and ordinary observer: $Q^{(k)}{}_{n} = (-1)^{k+n} W Q^{(n)}{}_{k}$.

Conflict of Interests

The authors have not declared any conflict of interests.

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Modeling and simulation of watershed erosion: Case study of Latian dam watershed

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The Latian Dam is one of the important drinking water resources of Tehran, and it also has a role in preventing the flood. It is very important to keep the quality of water and preventing the dam from filling. Considering the special aspects of RUSLE model, it was used for estimating the amount of erosion in the watershed of dam including the sub-basins: Jajrood River, Kond River and Afjeh River, and then compared with the actual values measured. The results of modeling show that the degree of erosion is high because of steep slopes, lack of plant coverage. The results of modeling the amount of erosion in The Jajrood basin have been estimated about 1,524 ton/year, in Kond basin about 228.5 ton/year, and in the Afjeh basin about 103.1 ton/year. By using the results of the water samples analyses, the amount of phosphorus entering the reservoir by the rivers were calculated. This research shows that, by using the proper coverage in the basin, the amount of sediment and phosphorus entering the reservoir, decreases considerably.

Key words: Latian Dam, phosphorus, reservoir.

INTRODUCTION

Soil erosion can cause ecological changes in the region. With regard to the role of planning and study of human and natural changes in erosion, assessment and adaptation model applied to each region, it is important. Latian Dam watershed is one of the areas where soil erosion is serious; the study of soil erosion in the area, as one of the water sources of Tehran is very important, particular area, population expansion, land use changed faces (Water and Energy Center of Sharif University, 2003a). Human sewage entering the river increased risk of erosion and sedimentation, pollution and nutrients in the reservoir. Including research done in this area; Quantitative modeling of soil erosion using AHP (Analytic hierarchy process) in the watershed Latian (Maleki et al., 2011), evaluate the accuracy and efficiency of computer models **II SEDIMOT** in estimating runoff and sediment (Sadeghi, 1994), and the application and model evaluation M.P.S.I.A.C. Using satellite imagery, geographic information systems (GIS) in the sub-basin Lavarak (Tahmasebipoor, 1995) and comparison of models RUSLE and SWAT to estimate Erosions in the sub-basin Amameh (Poorabdollah, 2007). In this study, RUSLE model helping GIS system for modeling soil erosion in the watershed Latian was used. And modeling results are

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Table 1. The data used in the model.

S/N	Model inputs	Applied data
1	Slope steepness and slope length	GIS map 1:50,000 Scale
2	Rainfall erosivity	Rainfall intensity got from Tehran's Regional Water Organization for 2 years
3	Meteorology	Temperature and rainfall Values at the different sub-basins
4	Soil erodibility	The soil map (1:50,000), separate reports including: the percentage of silt, sand, organic materials, soil structure and soil.
5	Land-use	1:50,000 maps containing the layers of orchards, pastures and farmlands with the related reports from Iranian National Geography Organization

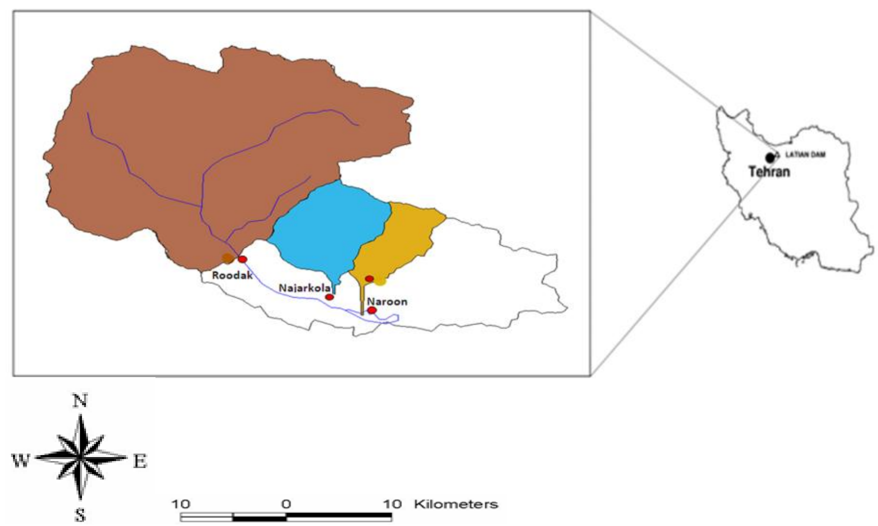

Figure 1. The location of stations and their upstream sub-basins.

compared with the actual values measured. The application of the RUSLE model has some advantages: (i) the data required are not very complex or unavailable in a developing country; (ii) this model is compatible properly with GIS software (Blonn, 2001), (iii) the use of this model is simplified by the presence of a graphical environment. Using this model with GIS information in raster format, the potential erosion can be found in any cell (Cox and Madramootoo, 1998). Also in regards to land use, based on management decisions, simulated erosion and its impact on the amount of sediment and phosphorus transport into the reservoir is shown. Study and modeling of the sub-basin Jajrood, and sub-basins kond and Afjeh is performed.

METHODS AND MATERIALS

The base equation of RUSLE model is as follows (Yazidhi, 2003):

$$A = LS \times R \times K \times C \times P$$

Where: A = the average annual soil loss (ton/ha/year); LS = the combination of the slope steepness and slope length (the factor without dimension); R = the rainfall erosivity factor; K = the factor of soil erodibility; C = the coefficient of plant coverage, and P = the coefficient of support practice.

The definition and application of every one of the above mentioned coefficients have been presented (Wischmeier and Smith, 1978; Desmet and Govers, 1996; Wischmeier et al., 1971).

The collected information about the watershed of Latian Dam is shown in Table 1. These stations together with their upstream sub-basins are shown in the Figure 1 and characterized in Table 2.

Input data

Figure 2 shows map of the river and its subdivisions, watershed boundaries and sub basins. Figure 3 shows the digital elevation map of the watershed (DEM) and the average slope values for each region. Figure 4 shows the type of vegetation in the watershed. Coefficients related to the vegetation which should be used in RUSLE model are shown in Table 3 and the values have been obtained from studies conducted in the region (Tehran's Agricultural Organization, 2002a).

Table 2. The specifications of measuring stations in the watershed of Latian dam.

Basin area (km^2)	Sub-Basin	Elevation (m)	Latitude	Longitude	River	Station
403	Garmabdar, Meygoon, Ahar, Emameh and Roodak	1700	35-53	51-32	Jajrood	Roodak
58	Kond	1670	35-49	51-38	Kond	Najarkola
31	Afjeh	1790	35-50	51-40	Afjeh	Naroon

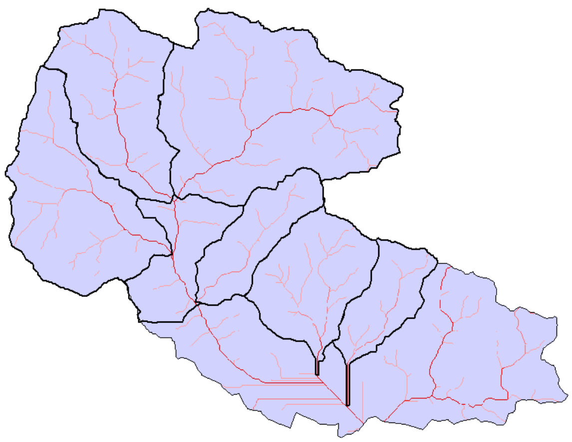

Figure 2. Distinguishing the boundaries of sub-basins with the help of arc view.

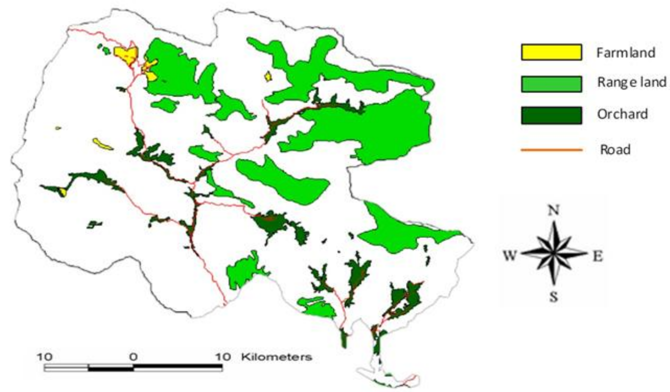

Figure 4. The map of plant.

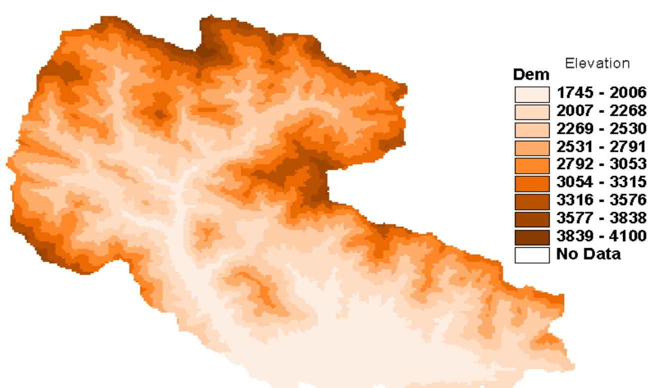

Figure 3. The DEM of the Latian Dam watershed.

The necessary inputs for the RUSLE model are: Average monthly temperature, the average amount of rainfall in month, the erosivity factor (R) and were obtained by studying the measured data in the stations (Tehran's Agricultural Organization, 2002b).

Soil types in this area include: Loamy, sandy loam, and clay loam. Types of Watershed soils are: 1 – Mountains; 2-hills; 3- flats and upper terraces; 4- Plains. Watershed land units, as part of the soil types are mentioned and Have the same physical characteristics, as of 1.1, 1.2, 1.3, 1.4, 2.1, 2.2, 3.1, 4.1 as shown in Figure 5 (Water and Energy center of Sharif University, 2003b).

The existing hydrological groups are shown in Table 4.

RESULTS AND DISCUSSION

The amounts of erosion, obtained from the model for any type of soil and any type of Land cover were presented for every sub-basin in Table 5. So, the whole amount of erosion per year in the basins of Roodak, Afjeh and Kond can be calculated by adding up the results of erosion in their sub-basins.

Due to the global equation erosion between any two regions with similar characteristics, whatever the place, the slope is greater or less vegetation or soil permeability is less, The amount of erosion in the area further. The result shows due to high slope and low vegetation in most areas, soil erosion is high.

Comparison with actual amounts and determining the model precision

A part of the eroded soil is transferred to downstream area by the flowing water in the form of sediment. This proportion is defined in the following way:

$$\text{Sediment delivery ratio (SDR)} = \frac{\text{The amount of sediment delivered to a point}}{\text{The amount of eroded soil at upstream of the point}}$$

To determine SDR and estimation of suspended load, the formulas proposed in this regard has been used (Foster, 2003).

Table 3. RUSLE input parameters for plant coverage.

Kind of plants	Canopy cover (%)	Falling height (m)	Residue	Canopy shape	Rock cover
Range land	50	0.2	Range litter	Rectangle	30
Farmland	30	0.3	The roots and branch residue	Rectangle	30
Orchard	35	2.1	Bushes and branches and leaves	Rectangle	30
barren land	-	-	-	-	30

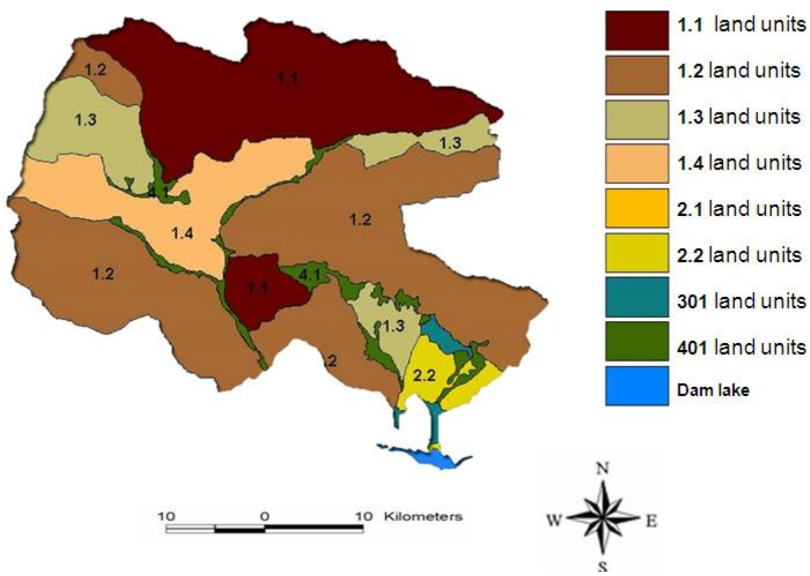

Figure 5. The map of evaluating the sources and the capability

Table 4. Hydrological groups of soil in the Latian Dam watershed.

Hydrological group	Minimum permeability (cm/h)	Runoff generation potential
A	7.5 - 11.5	Low
B	3.5 - 7.5	Low to moderate
C	1.5 - 3.5	Relatively high
D	< 1.5	High

The summary of the results (Including sediment calculated values and measured values and Model precision) are shown in Table 6.

Estimating the phosphorus load entering the water due to erosion

Total phosphate in the water consists of dissolved phosphorus and particulate phosphorus. The phosphorus existing in the sediments from soil erosion is particulate phosphorus. The amount of phosphorus in the unit of suspended sediment load is calculated from dividing particulate phosphorus load by suspended sediment load. This amount can be used for estimating the particulate phosphorus load entering the water after using the management procedures for preventing the erosion (the change of land-use) and investigating the change of the amount of phosphorus using these procedures. For calculating the phosphorus load, the results of the tests done by Sharif University of Technology were used. Figure 6 shows the variation of phosphorus-discharge at the mentioned stations.

The amounts of particulate phosphorus are shown in Table 7 based on the results of existing data.

Table 5. The results of erosion modeling in the sub-basins.

Sub-basin	The type of soil	Permeability (cm/h)	Land cover	Slope (degree)	Area (km^2)	Erosion (ton/ km^2/year)	Erosion (ton/year)
Garmabdar	SL (1.1)	1.5	P	23.7	25.8	3362	86833
			F-L	23.6	0.173	425.8	73.67
			B-L	26.7	52.8	4482	236844
	L (1.3)	5.5	P	22.2	10.0	3138	31376
			O	16.4	1.16	2465	2860
			B-L	19.8	3.49	3138	10950
	L (1.2)	2.5	P	23.7	23.6	3810	89953
			B-L	26.2	19.4	4931	95701
	SL (4.1)	5.5	O	18.1	0.76	2241	1703
			B-L	17.7	1.52	2241	3407
	SL (1.4)	2.5	O	15.8	0.871	2465	2147
			B-L	22.6	15.4	3586	55093
			P	27.9	2.75	3810	10481
Meygoon	SL (1.1)	1.5	F-L	18.3	1.94	336.2	650.8
			P	20.3	8.50	2913	24773
			B-L	25.7	21.5	4482	96531
	L (1.2)	2.5	F-L	20.7	0.331	358.6	118.7
			P	14.2	0.217	1860	403.7
			B-L	21.9	8.19	3586	29350
	L (1.3)	5.5	B-L	24.8	18.4	3362	61822
			O	18.5	0.600	2465	1479
	SL (4.1)	5.5	B-L	18.4	1.96	2465	4832
			O	11.5	1.23	1345	1655
	SL (1.4)	2.5	O	18.5	0.917	2914	2672
			F-L	24.6	0.283	403.4	114.2
			B-L	25.0	12.6	3810	48173
Ahar	L (1.3)	4	B-L	27.8	10.2	3810	38679
	SL (1.4)	5.5	B-L	25.2	24.7	3362	83169
			F-L	20.7	0.041	448.2	18.38
	SL (4.1)	5.5	B-L	19.1	1.57	2465	3878
			O	13.2	0.520	1569	815.8
	L (1.2)	2.5	O	14.8	1.76	2241	3940
			F-L	10.4	0.162	269	43.57
			B-L	23.9	55.2	3810	21085
Roodak	SL (1.4)	2.5	O	11.1	0.612	1591	973.8
			B-L	22.0	4.78	3586	17155
	SL (4.1)	5.5	O	10.4	0.290	1233	357.5
			B-L	22.4	1.79	3138	5626
	L (1.2)	2.5	P	22.5	1.19	3362	4000
			O	14.9	0.085	2465	210
			B-L	22.8	4.88	3810	18574
	SL (1.1)	1.5	B-L	27.2	7.58	4931	37378
	L (1.2)	2.5	O	5.7	0.212	672.3	142.5
			B-L	19.3	15.4	3138	48260
Emameh	L (1.2)	2.5	P	29.9	5.81	4483	26042
			O	26.8	0.472	4706	2221

Table 5. Contd.

			B-L	24.0	12.3	4706	57969
	SL (4.1)	5.5	O	8.9	2.18	1009	2197
			B-L	13.1	0.711	1726	12267
	SL (1.1)	2.5	O	16.1	0.073	2465	180.0
			B-L	22.2	8.35	3586	29952
	L (1.2)	2.5	P	26.6	2.82	4034	11380
			B-L	22.2	4.38	4482	19641
Afjeh	L (1.2)	2.5	O	22.0	0.307	3586	1101
			P	29.3	3.08	4258	13128
			B-L	25.3	15.0	4931	73894
	SL (3.1)	5.5	O	12.7	0.27	1569	423.6
			B-L	18.2	1.37	2465	3375
	SL (4.1)	5.5	O	10.2	1.68	1188	1990
			B-L	8.7	1.39	1031	1427
	CL (2.2)	2.5	B-L	12.5	4.53	1031	4672
			O	11.3	0.984	874.1	860.1
	CL (2.2)	2.5	B-L	14.7	1.68	1233	2075
			O	10.6	0.09	806.8	72.61
	SL (3.1)	5.5	O	2.30	0.34	150.2	51.50
			B-L	2.00	0.336	145.7	48.95
Kond	L (1.2)	2.5	P	31.3	6.46	4931	31827
			O	19.5	0.094	3362	316.00
			B-L	29.4	14.4	6051	87166
	SL (4.1)	5.5	O	11.0	1.24	1390	1716
			B-L	13.5	4.01	1860.	7465
	SL (3.1)	5.5	B-L	22.1	1.38	3138	4330
	L (1.3)	5.5	O	12.6	2.21	2062	4563
			B-L	13.5	8.5	2241	19050
	CL (2.2)	2.5	B-L	15.1	3.09	1345	4155
	L (1.2)	2.5	O	12.0	0.536	1927	1033
			P	22.8	1.93	3586	6921
			B-L	21.5	13.4	4482	59929

SL=Sandy loam, L= Loamy, CL= Clay loam; F= Farm land, Barren land, Pasture=P, Orchard=O.

Table 6. The results of calculated sediment load by the model and measurement.

Station	Soil loss (ton/year)	SDR (%)	Estimated total sediments (ton/year)	Calculated suspended load (ton/year)	Actual suspended load (ton/year)	Model precision (%)
Roodak	1524	16	243.8	187.6	216.0	86.8
Najarkola	228.5	24.7	56.49	43.38	50.09	86
Naroon	103.1	28	288.9	22.22	40.64	55*

*****In Afjeh because the lack of actual suspended load data, this number (40637) in the above table is not a good indicator for this parameter. Therefore the obtained precision is not a suitable value for model in Afjeh basin.

The change of land-uses and estimating the erosion variation

Now, with the change of the land-uses in the basin according to the Figure 7, erosion modeling was performed. The results of running the model again are presented in Table 8. The compared results are presented in Table 9.

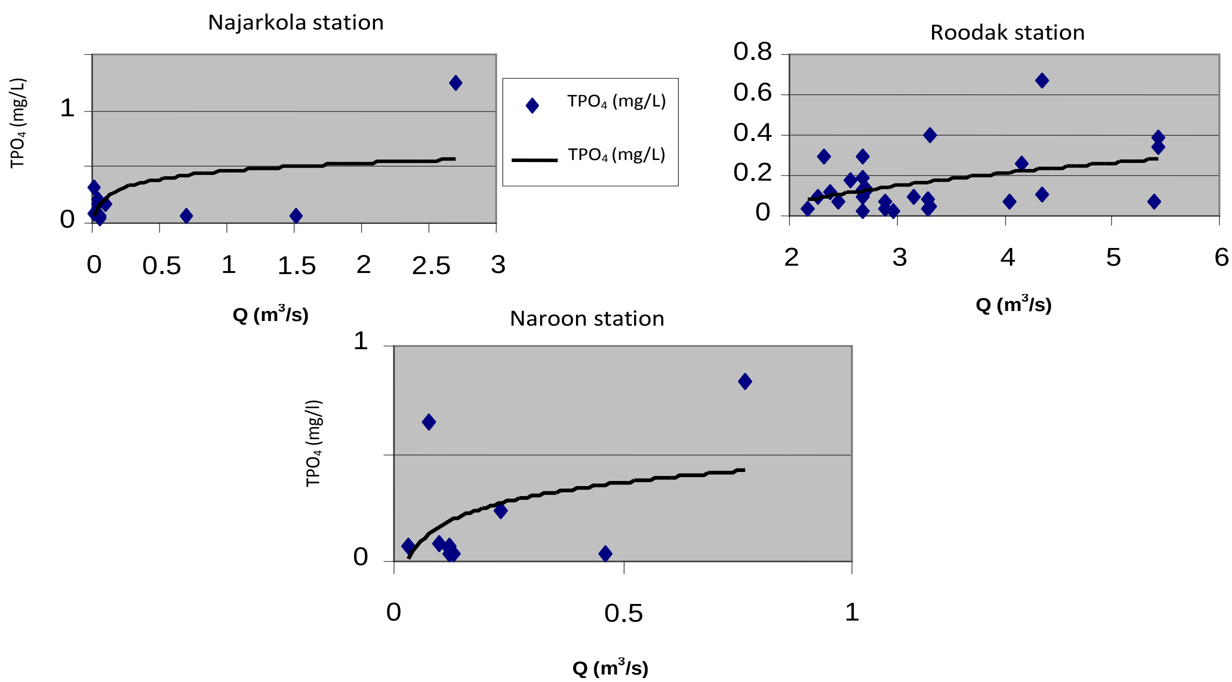

Figure 6. Variation of total phosphate (TPO_4) and river discharge (Q) at the Najarkola, Roodak and Naroon stations.

Table 7. The amounts of particulate phosphorus due to erosion in the stations.

Basin	The particulate phosphorus resulted from erosion (ton/year)	The amount of phosphorus in the suspended sediment load (g/ton)
Roodak	12.10	56.21
Kond	1.587	31.7
Afjeh	0.295	7.3*

* In Afjeh because the lack of actual suspended load data, this number in the above table is not a good indicator for this parameter.

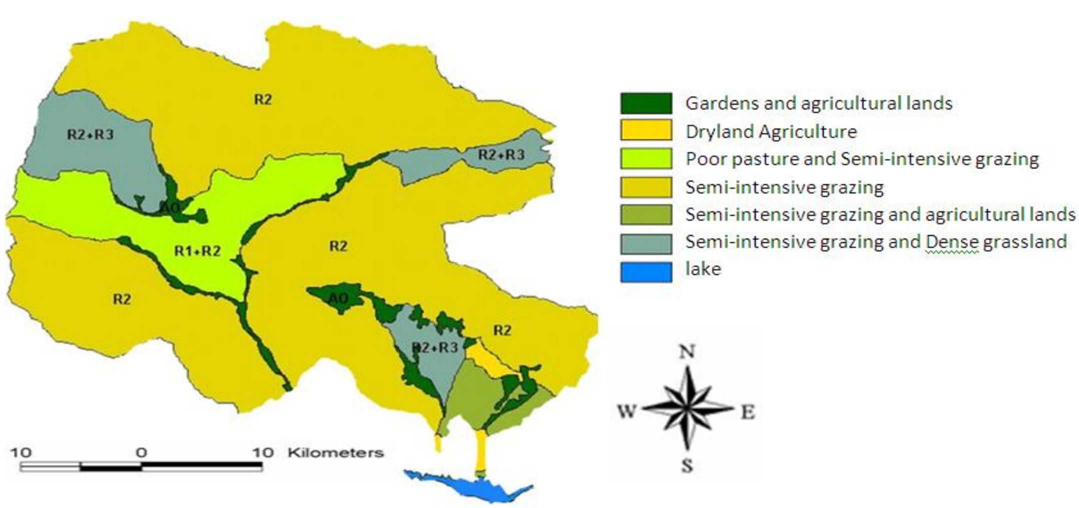

Figure 7. The map of present land use.

Table 8. The result of erosion modeling in the basin with new land.

Basin	Area	The recommended land-use	Slope (degree)	Area (km^2)	Erosion (ton/km^2 /year)	Erosion (ton/year)
Garmabdar	1.1	R2	25.98	78.90	1569	123700
	1.3	R2+R3	22.39	14.60	538	7864
	4.1	AO	15.58	2.28	1569	3773
	1.4	R1+R2	24.32	19.00	2465	46840
	1.2	R2	25.58	42.90	1726	73980
Meygoon	1.1	R2	25.42	31.90	1524	48554
	1.2	R2	21.10	8.73	1345	11739
	1.3	R2+R3	24.93	19.00	650	12348
	1.4	R1+R2	22.65	13.80	2174	30043
	4.1	AO	17.43	3.19	1905	6077
Ahar	1.2	R2	24.15	57.11	1905	108793
	1.3	R2+R3	28.08	10.15	1053	10691
	1.4	R1+R2	25.22	24.77	2689	66616
	4.1	AO	21.28	2.09	2465	5457
Roodak	1.1	R2	29.10	7.55	1927	14556
	1.2up	R2	23.03	6.15	1905	11721
	1.2down	R2	21.18	15.60	1726	26921
	1.4	R1+R2	22.50	5.41	2465	13344
	4.1	AO	20.26	2.08	2465	5428
Emameh	1.1	R2	23.73	8.46	1793	15161
	1.2up	R2	26.90	18.61	2241	41708
	1.2down	R2	24.97	7.15	2107	15054
	4.1	AO	12.25	2.89	1277	3688
Kond	1.2up	R2	29.78	20.96	2465	51672
	1.2down	R2	21.33	15.86	1748	27725
	4.1	AO	12.88	5.25	1412	7413
	1.3	R2+R3	13.25	10.71	426	4562
	3.1	DF	21.93	1.37	538	739
	2.2	R2+F	15.10	3.06	359	1098
Afjeh	1.2	R2	25.91	18.37	1121	20582
	4.1	AO	9.43	3.06	874	2674
	2.2up	R2+F	12.02	5.53	269	1488
	2.2down	R2+F	14.55	1.75	336	588
	3.1up	DF	16.72	1.65	381	627
	3.1down	DF	2.49	0.679	29.1	19.8

Table 9. The comparison of the results of erosion modeling in the present condition with the suggested land-use conditions.

Basin	The erosion with new land-use (ton/year)	The erosion with the former land-use (ton/year)	The calculated suspended load with new land-use (ton/year)	The percentage of erosion reduction	The particulate phosphorus entering after the new land-use (ton/year)
Roodak	703.5	1524	86.58	54	4.87
Kond	93.21	228.5	17.71	59	0.56
Afjeh	25.98	103.1	5.595	75	0.041

Conclusion

By comparing the results of sediment load calculated by the model with the actual values, the precision of the model in estimating the erosion and the sediment yield for the upstream basins of Roodak, Kond and Afjeh was respectively of 86.8, 86 and 55%, although depend on the percentage assumed valid for SDR. These results show the suitable precision of the model for Roodak and Kond basins (if in the basin of Afjeh, more measuring data for suspended load could be gathered, the more precise outputs of the model can be obtained in this basin).

The results show that the use of vegetation to areas without coverage and use of appropriate vegetation density appropriate land, the amount of erosion and the phosphorus load are reduced considerably. As a result, by using the suggested methods of land-uses for the basins discussed so far, the degree of erosion reduction in the upper basins of Roodak is about 54%, for Kond basin is about 59% and in the basin of Afjeh is about 75%.

Conflict of Interests

The author(s) have not declared any conflict of interests.

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